

A New Operator Theory Similar to Pseudo-Differential Operators

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Abstract

We summarize and extend the correlative definitions and principles of abstract operators, discuss the relation between abstract operators and pseudo-differential operators, add several new algorithms, furthermore, develop the theory of partial differential equations with abstract operators, and then systematically expound the basic methods. By combining abstract operators with the Laplace transform, we can easily derive the explicit solution of initial value problem of linear higher-order partial differential equations for n-dimensional space, and establish the general theory of linear higher-order partial differential equations.

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1 Introduction

In 1960s, the general theory of linear partial differential equations made important progress by using the generalized function and its Fourier transform, further, the theory of pseudo-differential operators. The significance of the latter theory lies in introducing the concept of symbol, but due to the lack of awareness of the analytic continuous fundamental theorem (See [1]), the concept of abstract operators remained unclear. Without the theories of abstract operators, the operator method of differential equations is just a technique, and the solution of partial differential equations is normally complex. In 1997, the author delivered the analytic continuous fundamental theorem, based on which the author introduced the concept of abstract operators, and derived the algorithms of five types of abstract operators (See [1]) such as

$$\exp(h\partial_x), \sin(h\partial_x), \cos(h\partial_x), \sinh(h\partial_x) \text{ and } \cosh(h\partial_x).$$

Where $\cos(ih\partial_x) = \cosh(h\partial_x)$, $\sin(ih\partial_x) = i \sinh(h\partial_x)$. Obviously, the abstract operator uses the same notation as infinite order differential operator, as it is the extension of the latter. However, within the frame of infinite order differential operators, complex operators are difficult to define, such as

$$\exp(tP(\partial_x)), \cos(tP(\partial_x)^{1/2}), \frac{\sin(tP(\partial_x)^{1/2})}{P(\partial_x)^{1/2}}, \cosh(tP(\partial_x)^{1/2}), \frac{\sinh(tP(\partial_x)^{1/2})}{P(\partial_x)^{1/2}}.$$

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But as abstract operators, we can easily establish their algorithms (See [1], [2]), therefore the general solving procedure of initial value problem of linear partial differential equations with constant coefficients is derived [1], [2], [3], this approach can also be applied in solving the partial differential equations with variable coefficients containing t .

According to [4], pseudo-differential operators are defined as:

If $a(x, \xi) \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$, and for arbitrary $\alpha, \beta \in \mathbb{N}^n$ and an real number m ,

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m - |\alpha|}$$

is tenable, where $C_{\alpha, \beta}$ is a constant, then the linear continuous mapping A of $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ can be defined as:

$$Au(x) = (2\pi)^{-n} \int e^{i\xi x} a(x, \xi) \hat{u}(\xi) d\xi, \quad (1)$$

which are called the pseudo-differential operators, denoted by $a(x, D)$, where $a(x, \xi)$ are the symbols of $a(x, D)$, D denotes

$$D^\alpha = \left(\frac{1}{i} \partial_{x_1} \right)^{\alpha_1} \left(\frac{1}{i} \partial_{x_2} \right)^{\alpha_2} \cdots \left(\frac{1}{i} \partial_{x_n} \right)^{\alpha_n}.$$

Obviously, this definition means the following equation is tenable, namely

$$a(x, D)e^{i\xi x} = a(x, \xi)e^{i\xi x}. \quad (2)$$

Therefore pseudo-differential operators are similar to abstract operators. As the definition formula of pseudo-differential operators is a Fourier integral expression, their symbols are largely restricted to ensure the convergence of Fourier integrals. But for many specific problems, the symbols of pseudo-differential operators should not be restricted that much. Thus, a better definition of symbols can be expressed as:

$$\sigma_A(x, \xi) = e^{-i\xi x} A e^{i\xi x}. \quad (3)$$

Where A are properly extended pseudo-differential operators. Certainly, this extension still has its limitations as it involves another type of Fourier integrals.

Then can we get rid of Fourier integrals when defining pseudo-differential operators? Let us study the following expression:

$$A e^{i\xi x} = \sigma_A(x, \xi) e^{i\xi x}. \quad (4)$$

Obviously, the properties of A depend on $\sigma_A(x, \xi)$. (4) indicates the mapping relation between A and $\sigma_A(x, \xi)$. For a given function $\sigma_A(x, \xi)$, if the corresponding algorithms of A can be derived from this type of mapping relations, to determine the domain and range of A , then this type of mapping relations can be the best definition of pseudo-differential operators. Luckily, this idea totally works by using the analytic continuous fundamental theorem or the abstract operator fundamental theorem. The pseudo-differential operators established by using this method are the abstract operators.

Now let us carry out a systematic review of the basic methods of abstract operators, and prove that if we combine abstract operators with the Laplace transform on $C^\infty(\mathbb{R}_x^n \times \mathbb{R}_t^1)$, then no matter how complex the initial value problem of linear partial differential equations is, the solving process is simply easy. In addition, in order to achieve self consistency theoretically, it is necessary to derive rules of differentiation from the view of abstract operators.

2 Basic theory of abstract operators

2.1 Basic concepts, research method and basic operations

Definition 1. (See [1]) Within each convergence circle of analytic functions, if the effects on the series term by term from the linear operator converge uniformly to the effects on the sum function, then the operator is called having the analytic continuity.

Definition 2. The linear continuous mapping A of $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ has the Fourier continuity, if A acts on an arbitrary function $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$Af(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi) A e^{i\xi x} d\xi.$$

Where the Fourier transform $\hat{f} = \mathcal{F}f$ is defined as

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\xi x} dx, \quad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

The Fourier transform \mathcal{F} is an isomorphic mapping in the space $\mathcal{S}(\mathbb{R}^n)$, and its inverse transform is

$$\mathcal{F}^{-1} \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi x} d\xi = f(x).$$

Definition 3. (See [1]) x^α , $x \in \mathbb{R}^n$ is called the base function, its exponential form is $e^{\xi x}$, where $\alpha \in \mathbb{N}^n$, $\xi \in \mathbb{R}^n$ or \mathbb{C}^n are respectively called the character of x^α and $e^{\xi x}$.

Definition 4. (See [1]) A class of linear operators is called the abstract operators, denoted by $f(t, \partial_x)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}^1$, and if it acts on the exponential base function $e^{\xi x}$, we have

$$f(t, \partial_x) e^{\xi x} = f(t, \xi) e^{\xi x}, \quad \forall f(t, \xi) \in C^\infty(\mathbb{R}_t^1 \times \mathbb{R}_\xi^n).$$

Which is called the abstract operators taking ∂_x as the operator element, where $f(t, \xi)$ is called the symbol of the abstract operators $f(t, \partial_x)$.

If abstract operators $f(t, \partial_x)$ is a linear operator having the Fourier continuity, then it similar to a pseudo-differential operators.

Definition 5. A class of linear operators is called the abstract operators, denoted by $f(t, x \frac{\partial}{\partial x})$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}^1$, where

$$x \frac{\partial}{\partial x} = \left(x_1 \frac{\partial}{\partial x_1}, x_2 \frac{\partial}{\partial x_2}, \dots, x_n \frac{\partial}{\partial x_n} \right),$$

and if it acts on the base function x^α , we have

$$f\left(t, x \frac{\partial}{\partial x}\right) x^\alpha = f(t, \alpha) x^\alpha, \quad \forall f(t, \alpha) \in C^\infty(\mathbb{R}_t^1 \times \mathbb{N}^n).$$

Which is called the abstract operators taking $x \frac{\partial}{\partial x}$ as the operator element, where $f(t, \alpha)$ is called the symbol of the abstract operators $f(t, x \frac{\partial}{\partial x})$.

Definition 6. We call any operator element and common variables the abstract element, denoted by X, Y, \dots and $X = (X_1, X_2, \dots, X_n)$, $Y = (Y_1, Y_2, \dots, Y_n)$, etc.; Accordingly, $f(X)$,

$f(Y)$, etc., represent the abstract operators or common functions, which is called the operator function.

Abstract operators fundamental theorem. Let A, B, A', B' be the abstract operators, if there are functions $v(t) \in \mathcal{D}(A'), u(t) \in \mathcal{D}(B'), t \in \mathbb{R}^1$ and abstract elements X, Y , making one of the following two expressions tenable

$$AX^\alpha A'v(t) = BY^\alpha B'u(t) \quad \text{or} \quad Ae^{\xi X} A'v(t) = Be^{\xi Y} B'u(t),$$

and the expressions of A, B, A', B' do not explicitly contain the character $\alpha \in \mathbb{N}^n, \xi \in \mathbb{R}_n$ or \mathbb{C}_n , then

$$Af(X)A'v(t) = Bf(Y)B'u(t), \quad \forall f(X), f(Y) \in \mathcal{D}(A) \cap \mathcal{D}(B).$$

Where $\mathcal{D}(\cdot), \mathcal{R}(\cdot)$ are domains and ranges of operators. $\mathcal{D}(f(X)) = \mathcal{R}(A'), \mathcal{D}(f(Y)) = \mathcal{R}(B'), \mathcal{D}(A) = \mathcal{R}(f(X)), \mathcal{D}(B) = \mathcal{R}(f(Y))$.

When $A'v(t) = B'u(t) = I$, and $X = x \in \mathbb{R}^n, Y = y \in \mathbb{R}^n$, this theorem becomes the following **Analytic continuous fundamental theorem:**

If there are functions $x(t), y(t) \in \mathbb{R}^n, t \in \mathbb{R}^1$, making one of the following two expressions tenable

$$Ax^\alpha = By^\alpha \quad \text{or} \quad Ae^{\xi x} = Be^{\xi y},$$

and the expressions of abstract operators A, B do not explicitly contain the character $\alpha \in \mathbb{N}^n, \xi \in \mathbb{R}_n$ or \mathbb{C}_n , then

$$Af(x) = Bf(y), \quad \forall f(x), f(y) \in \mathcal{D}(A) \cap \mathcal{D}(B).$$

Especially, when A, B are the linear operators having the analytic continuity, the analytic continuous fundamental theorem has been proved by G.Q Bi in 1997 (See [1]).

The abstract operators fundamental theorem is the fundamental property of abstract operators, which directly extends the domains of abstract operators from power functions and exponential functions to a wider range of functions.

Definition 7. A linear operator equation, if it's true for arbitrary functions or operator functions in a certain range, then it is called an operator formula. Particularly, the operator formulas that determine the domain and range of abstract operators are called the algorithm of the abstract operators.

Definition 8. The relational expression between each component of the character $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ or $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ is called the characteristic equation. In general, an operator formula becomes the corresponding characteristic equation, when where arbitrary function of which is the base function.

Corollary 1. As the base function can be expressed by both power function and exponential function, a single operator formula can be corresponding to two different characteristic equations, conversely, a single characteristic equation can also be corresponding to two different operator formulas.

Constructing the operator equation tenable for the base function by the characteristic equation through the definition of abstract operators, and then deducing that it is also tenable for arbitrary functions or operator functions in a certain range by using the analytic continuous fundamental

theorem or the abstract operators fundamental theorem, thus we derive new operator formulas. Finding or establishing a new operator formula requires the knowledge of the corresponding characteristic equation in advance, without knowing the specific form of the new operator formula. Therefore, it all boils down to seek or construct appropriate characteristic equations. The key to transform the characteristic equation to the corresponding operator formula, is constructing the operator equation true for the base function by using the specific form of the characteristic equation and the definition of abstract operators, and also, only when the operator constructed is linear and the expression of the linear operator doesn't explicitly contain the character of the base function can we derive that the operator equation is not only true for the base function, but also for arbitrary functions in a certain range, according to the analytic continuous fundamental theorem or the abstract operators fundamental theorem. This is the research method of abstract operators.

Example 1. Let $x \in \mathbb{R}^n$, $\xi \in \Omega \subseteq \mathbb{R}_n$, then $\forall f(\xi) \in C^\infty(\Omega)$ we have

$$\begin{aligned} f(\partial_x) \sin bx &= \sin \left(bx + \frac{\pi}{2} \xi_1 \frac{\partial}{\partial \xi_1} + \frac{\pi}{2} \xi_2 \frac{\partial}{\partial \xi_2} + \cdots + \frac{\pi}{2} \xi_n \frac{\partial}{\partial \xi_n} \right) f(\xi) \Big|_{\xi=b}, \\ f(\partial_x) \cos bx &= \cos \left(bx + \frac{\pi}{2} \xi_1 \frac{\partial}{\partial \xi_1} + \frac{\pi}{2} \xi_2 \frac{\partial}{\partial \xi_2} + \cdots + \frac{\pi}{2} \xi_n \frac{\partial}{\partial \xi_n} \right) f(\xi) \Big|_{\xi=b}. \end{aligned} \quad (5)$$

Where $bx = b_1x_1 + b_2x_2 + \cdots + b_nx_n$.

Proof. Selecting the following two expressions from the known formulas as the characteristic equations, namely

$$\partial_x^\alpha \sin bx = b^\alpha \sin \left(bx + \frac{\pi}{2} |\alpha| \right) \quad \text{and} \quad \partial_x^\alpha \cos bx = b^\alpha \cos \left(bx + \frac{\pi}{2} |\alpha| \right).$$

Taking $\alpha \in \mathbb{N}^n$ as the character of the base function, ∂_x and $\xi \frac{\partial}{\partial \xi}$ as the operator elements, we have the characteristic equation with the following two operator equations by Definition 5:

$$\begin{aligned} \partial_x^\alpha \sin bx &= \sin \left(bx + \frac{\pi}{2} \xi_1 \frac{\partial}{\partial \xi_1} + \frac{\pi}{2} \xi_2 \frac{\partial}{\partial \xi_2} + \cdots + \frac{\pi}{2} \xi_n \frac{\partial}{\partial \xi_n} \right) \xi^\alpha \Big|_{\xi=b}, \\ \partial_x^\alpha \cos bx &= \cos \left(bx + \frac{\pi}{2} \xi_1 \frac{\partial}{\partial \xi_1} + \frac{\pi}{2} \xi_2 \frac{\partial}{\partial \xi_2} + \cdots + \frac{\pi}{2} \xi_n \frac{\partial}{\partial \xi_n} \right) \xi^\alpha \Big|_{\xi=b}. \end{aligned}$$

According to the abstract operators fundamental theorem, we get the Example 1.

Definition 9. Let A be a linear operator having the analytic continuity or Fourier continuity, if there is another linear operator, denoted by A^{-1} , making $AA^{-1} = A^{-1}A = I$, then A^{-1} is called the inverse operator of A .

By Definition 4 we have the following Corollary:

Corollary 2. (See [1]) The operator algebras formed by all the abstract operators $f(t, \partial_x)$, are isomorphic to the algebras formed by their symbols $f(t, \xi)$. This isomorphism is determined by $f(t, \partial_x) \leftrightarrow f(t, \xi)$, and

$$f(t, \partial_x) \pm g(t, \partial_x) \leftrightarrow f(t, \xi) \pm g(t, \xi), \quad f(t, \partial_x) \circ g(t, \partial_x) \leftrightarrow f(t, \xi)g(t, \xi).$$

Especially, the abstract operators $f(\partial_x)$ and $g(\partial_x)$, $x \in \mathbb{R}^n$ are each other's inverse operator, if and only if their symbols $f(\xi)$ and $g(\xi)$ satisfy $f(\xi)g(\xi) = 1$, $\xi \in \mathbb{R}_n$.

For instance, $\partial_{x_i} \sin(x\partial_y) = \partial_{y_i} \cos(x\partial_y)$, $\partial_{x_i} \cos(x\partial_y) = -\partial_{y_i} \sin(x\partial_y)$, where $x, y \in \mathbb{R}^n$.

Corollary 3. (See [1]) Let $g(x) \in C^\infty(\mathbb{R}^n)$, if $g(\partial_\xi)(e^{\xi x} f(\xi))$ is continuous at $\xi = a$, then

$$f(\partial_x)(e^{ax} g(x)) = g(\partial_\xi)(e^{\xi x} f(\xi))|_{\xi=a}. \quad (6)$$

Obviously, by Definition 5 we can also get similar Corollary.

Example 2. If function $f(x)$, $x \in \mathbb{R}^n$ satisfies

$$\int_{\mathbb{R}^n} f(\eta) \exp\left(-\frac{|\eta - 2\alpha x|^2}{4\alpha}\right) d\eta < +\infty, \quad \eta \in \mathbb{R}^n, \alpha > 0,$$

then

$$f(\partial_x)e^{\alpha|x|^2} = e^{\alpha|x|^2} \frac{1}{2^n(\alpha\pi)^{n/2}} \int_{\mathbb{R}^n} f(\eta) \exp\left(-\frac{|\eta - 2\alpha x|^2}{4\alpha}\right) d\eta. \quad (7)$$

Proof. In the Corollary 3, let $g(x) = e^{\alpha|x|^2} \in C^\infty(\mathbb{R}^n)$ and $a = 0$, we have

$$\begin{aligned} f(\partial_x)e^{\alpha|x|^2} &= \exp(\alpha|\partial_\xi|^2)(e^{\xi x} f(\xi))|_{\xi=0} = \exp(\alpha|x + \partial_\xi|^2)f(\xi)|_{\xi=0} \\ &= e^{\alpha|x|^2} \exp(2\alpha x \partial_\xi) \exp(\alpha|\partial_\xi|^2)f(\xi)|_{\xi=0} \\ &= e^{\alpha|x|^2} \frac{1}{2^n(\alpha\pi)^{n/2}} \exp(2\alpha x \partial_\xi) \int_{\mathbb{R}^n} f(\eta) \exp\left(-\frac{|\eta - \xi|^2}{4\alpha}\right) d\eta \Big|_{\xi=0} \\ &= e^{\alpha|x|^2} \frac{1}{2^n(\alpha\pi)^{n/2}} \int_{\mathbb{R}^n} f(\eta) \exp\left(-\frac{|\eta - \xi - 2\alpha x|^2}{4\alpha}\right) d\eta \Big|_{\xi=0}, \quad \alpha > 0. \end{aligned}$$

Where $x\partial_\xi = x_1\partial_{\xi_1} + x_2\partial_{\xi_2} + \cdots + x_n\partial_{\xi_n}$, $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$.

Clearly, by (7) the $f(\eta)$ can also be a bounded measurable function. Therefore, Corollary 3 is an important way to extend the symbols and domains of abstract operators. In fact, limitations on the symbol $f(t, \xi)$ and the operator element X will be broadened, and more general abstract operators will be introduced to deal with problems involved in our further discussion.

There are two major research methods of the operator theory. One determines the domain and range of an operator first, then derives its operator algebra. The other abstractly determines an operator algebra, then derives the domain and range of the operator. The former is applied by pseudo-differential operators, while the latter is applied by abstract operators. The definition of abstract operators actually determines a kind of excellent operator algebras, then establishes different algorithms for different operators using the abstract operator fundamental theorem, further, derives different domains and ranges for abstract operators with different symbols. In other words, the domain and range of an abstract operator depend on the algorithm it applies. Therefore, a significant feature of abstract operators is focusing on the research of algorithms, without depending on the Fourier integral transform. The basic theory of abstract operators is a theory about algorithms, which has already become an important tool to seek and calculate explicit solutions of linear higher-order partial differential equations.

2.2 Algorithms of Abstract operators on differentiable function

In order to achieve theoretical rigor, firstly let us derive the rules of differentiation from the view of abstract operators, making the concept of ordinary or partial differential operator on the basis of the abstract operators itself.

Differential operators can be defined as the following abstract operators, namely

$$x \frac{d}{dx} x^\alpha = \alpha x^\alpha, \quad \frac{d}{dx} e^{\xi x} = \xi e^{\xi x}, \quad \forall x \in \mathbb{R}^1.$$

Where $\alpha \in \mathbb{N}^1$, $\xi \in \mathbb{R}_1$ are the symbols of abstract operators $x \frac{d}{dx}$ and $\frac{d}{dx}$ respectively.

Let a and b be the characters of base functions, as the characteristic equation, the binomial formula of integer power can be expressed as

$$(a + b)^n = \sum_{j=0}^n \binom{n}{j} a^j b^{n-j}, \quad \forall n \in \mathbb{N}.$$

By base functions e^{ax} and e^{bx} we can combine the binomial formula with the following operator equations:

$$\frac{d^n}{dx^n} (e^{ax} \cdot e^{bx}) = \sum_{j=0}^n \binom{n}{j} \frac{d^j}{dx^j} e^{ax} \cdot \frac{d^{n-j}}{dx^{n-j}} e^{bx}.$$

According to the analytic continuous fundamental theorem, it is tenable, namely

$$\frac{d^n}{dx^n} (vu) = \sum_{j=0}^n \binom{n}{j} \frac{d^j v}{dx^j} \frac{d^{n-j} u}{dx^{n-j}}, \quad \forall v, u \in C^n(\mathbb{R}^1). \quad (8)$$

This is the Leibniz rule. Similarly we have

$$v \frac{d^n u}{dx^n} = \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{d^{n-j}}{dx^{n-j}} \left(u \frac{d^j v}{dx^j} \right), \quad \forall v, u \in C^n(\mathbb{R}^1). \quad (9)$$

$$\frac{d^k}{dx^k} f(x^2) = \sum_{j=0}^{[k/2]} \frac{k!}{j! (k-2j)!} (2x)^{k-2j} \frac{d^{k-j}}{dy^{k-j}} f(y), \quad \forall f(x^2) \in C^k(\mathbb{R}^1), y = x^2. \quad (10)$$

By (10) we easily obtain the Hermite polynomials

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2} = \sum_{j=0}^{[k/2]} (-1)^j \frac{k!}{j! (k-2j)!} (2x)^{k-2j}.$$

The major parts of rules of differentiation are the derivative principle of function product, the derivative principle of compound function and the chain rule of multivariate function. Firstly let us establish the derivative principle of function product. Without losing the universality, we only consider the situation of the product of two functions. Let $n = 1$ in (8), then for $v = f(x)$ and $u = g(x)$ we have

$$\frac{d}{dx} (f(x)g(x)) = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x), \quad \forall f, g \in C^1(\mathbb{R}^1). \quad (11)$$

Let $\varphi(x) = f_1(x)f_2(x) \cdots f_n(x)$, generally we have

$$\frac{d}{dx} \varphi(x) = \sum_{j=1}^n \prod_{\substack{i=1 \\ i \neq j}}^n f_i(x) \frac{d}{dx} f_j(x), \quad \forall f_j(x) \in C^1(\mathbb{R}^1).$$

Let $f_1(x) = f_2(x) = \cdots = f_n(x) = g(x) = y$, then for differentiable function $y = g(x)$ we have

$$\frac{d}{dx}y^n = ny^{n-1}\frac{dy}{dx} \quad \text{or} \quad \frac{d}{dx}y^n = \frac{dy}{dx}\frac{d}{dy}y^n, \quad \forall n \in \mathbb{N}^1.$$

According to the analytic continuous fundamental theorem, we have the derivative principle of compound function $f(g(x)) \in C^1(\mathbb{R}^1)$:

$$\frac{d}{dx}f(g(x)) = \frac{dy}{dx}\frac{d}{dy}f(y), \quad y = g(x), \quad x \in \mathbb{R}^1. \quad (12)$$

By using (11) and (12), for differentiable functions $x_1(t)$ and $x_2(t)$, $t \in \mathbb{R}^1$ we have

$$\frac{d}{dt}(x_1^{\alpha_1}x_2^{\alpha_2}) = \frac{dx_1^{\alpha_1}}{dx_1}\frac{dx_1}{dt}x_2^{\alpha_2} + x_1^{\alpha_1}\frac{dx_2^{\alpha_2}}{dx_2}\frac{dx_2}{dt}, \quad \alpha_1, \alpha_2 \in \mathbb{N}.$$

Taking this one as the characteristic equation, by base functions x^α , $x \in \mathbb{R}^2$, $\alpha \in \mathbb{N}^2$, combining it with the following operator equation:

$$\frac{d}{dt}x^\alpha = \frac{dx_1}{dt}\frac{\partial}{\partial x_1}x^\alpha + \frac{dx_2}{dt}\frac{\partial}{\partial x_2}x^\alpha, \quad x(t) \in \mathbb{R}^2 \quad t \in \mathbb{R}^1 \quad \alpha \in \mathbb{N}^2.$$

According to the analytic continuous fundamental theorem, it is tenable $\forall f(x) \in C^1(\mathbb{R}^2)$, namely

$$\frac{d}{dt}f(x) = \frac{dx_1}{dt}\frac{\partial}{\partial x_1}f(x) + \frac{dx_2}{dt}\frac{\partial}{\partial x_2}f(x), \quad x(t) \in \mathbb{R}^2 \quad t \in \mathbb{R}^1.$$

Clearly, for $x(t) \in \mathbb{R}^n$, $t \in \mathbb{R}^1$, we can generally derive $\forall f(x) \in C^1(\mathbb{R}^n)$

$$\frac{d}{dt}f(x) = \frac{dx_1}{dt}\frac{\partial}{\partial x_1}f(x) + \frac{dx_2}{dt}\frac{\partial}{\partial x_2}f(x) + \cdots + \frac{dx_n}{dt}\frac{\partial}{\partial x_n}f(x). \quad (13)$$

If taking $n \in \mathbb{N}$ as the character of base function, the binomial formula can be expressed as the following characteristic equation:

$$(x+h)^n = \sum_{j=0}^{\infty} \frac{h^j}{j!} \frac{d^j}{dx^j}x^n.$$

According to the analytic continuous fundamental theorem, it is tenable for any analytic function $f(x) \in C^\infty(\mathbb{C}^1)$, namely

$$f(x+h) = \sum_{j=0}^{\infty} \frac{h^j}{j!} \frac{d^j}{dx^j}f(x), \quad x \in \mathbb{C}^1. \quad (14)$$

This is the Taylor formula. If taking (14) as a characteristic equation, then similarly we have

Theorem 1. If $\exists v(s), u(s) \in C^\infty(\mathbb{C}^1)$ can make the infinite series on the right side of (15) uniform convergent, then it will uniform converges to the left side of the formula, namely

$$f\left(\lambda \frac{\partial}{\partial s}\right)(vu) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{\partial^k v}{\partial s^k} f^{(k)}\left(\lambda \frac{\partial}{\partial s}\right)u, \quad s \in \mathbb{C}^1, |\lambda| \leq 1. \quad (15)$$

Corollary 4. In (15), when $\lambda = -1$, $v = s$, $u = 1/s$, $s \neq 0$, we have

$$f'\left(-\frac{\partial}{\partial s}\right)\frac{1}{s} = sf\left(-\frac{\partial}{\partial s}\right)\frac{1}{s} - f(0) \quad \text{or} \quad \mathcal{L}f'(t) = s\mathcal{L}f(t) - f(0). \quad (16)$$

Where $\mathcal{L}f(t)$ denotes

$$\mathcal{L}f(t) = f\left(-\frac{\partial}{\partial s}\right)\frac{1}{s}, \quad \Re(s) > 0, \quad t \in \overline{\mathbb{R}_+^1}. \quad (17)$$

This is the internal connection between abstract operators and Laplace transform, thus the abstract operator becomes a significant tool to compute Laplace transform. For instance, by (17) we have

$$\mathcal{L}[g(t)f(t)] = g\left(-\frac{\partial}{\partial s}\right)F(s), \quad (F(s) = \mathcal{L}f(t)). \quad (18)$$

Similar to (11)-(13), Guangqing Bi has already established a series of algorithms of abstract operators in reference [1]:

$$\cos(h\partial_x)f(x) = \Re[f(x+ih)], \quad \sin(h\partial_x)f(x) = \Im[f(x+ih)], \quad \forall f(z) \in C^\infty(\mathbb{C}^n). \quad (19)$$

Where $bx = b_1x_1 + b_2x_2 + \cdots + b_nx_n$, $bh = b_1h_1 + b_2h_2 + \cdots + b_nh_n$.

$$\begin{aligned} \sin(h\partial_x)(uv) &= \cos(h\partial_x)v \cdot \sin(h\partial_x)u + \sin(h\partial_x)v \cdot \cos(h\partial_x)u, \\ \cos(h\partial_x)(uv) &= \cos(h\partial_x)v \cdot \cos(h\partial_x)u - \sin(h\partial_x)v \cdot \sin(h\partial_x)u. \end{aligned} \quad (20)$$

$$\begin{aligned} \sin(h\partial_x)\frac{u}{v} &= \frac{\cos(h\partial_x)v \cdot \sin(h\partial_x)u - \sin(h\partial_x)v \cdot \cos(h\partial_x)u}{(\cos(h\partial_x)v)^2 + (\sin(h\partial_x)v)^2}, \\ \cos(h\partial_x)\frac{u}{v} &= \frac{\cos(h\partial_x)v \cdot \cos(h\partial_x)u + \sin(h\partial_x)v \cdot \sin(h\partial_x)u}{(\cos(h\partial_x)v)^2 + (\sin(h\partial_x)v)^2}. \end{aligned} \quad (21)$$

Where $x \in \mathbb{R}^n$, $h \in \mathbb{R}_n$ or \mathbb{C}_n .

$$\begin{aligned} \sin(h\partial_t)f(x(t)) &= \sin(Y\partial_X)f(X), \\ \cos(h\partial_t)f(x(t)) &= \cos(Y\partial_X)f(X). \end{aligned} \quad x(t) \in \mathbb{R}^n, \quad t \in \mathbb{R}^1, \quad h \in \mathbb{R}_1 \text{ or } \mathbb{C}_1. \quad (22)$$

Where $X = (X_1, X_2, \cdots, X_n)$, $X_j = \cos(h\partial_t)x_j(t)$, $Y \in \mathbb{R}_n$, $Y_j = \sin(h\partial_t)x_j(t)$.

For example, when $n = 1$, the (22) can easily be restated as

$$\begin{aligned} \sin\left(h\frac{d}{dt}\right)f(x(t)) &= \sin\left(Y\frac{\partial}{\partial X}\right)f(X), \\ \cos\left(h\frac{d}{dt}\right)f(x(t)) &= \cos\left(Y\frac{\partial}{\partial X}\right)f(X). \end{aligned} \quad (23)$$

Where $Y = \sin(h\partial_t)x(t)$, $X = \cos(h\partial_t)x(t)$, $t \in \mathbb{R}^1$, $h \in \mathbb{R}_1$ or \mathbb{C}_1 .

If $n = 2$, then (22) can easily be restated as

$$\begin{aligned} \sin\left(h\frac{d}{dt}\right)f(x(t), y(t)) &= \sin\left(Y_x\frac{\partial}{\partial X_x} + Y_y\frac{\partial}{\partial X_y}\right)f(X_x, X_y), \\ \cos\left(h\frac{d}{dt}\right)f(x(t), y(t)) &= \cos\left(Y_x\frac{\partial}{\partial X_x} + Y_y\frac{\partial}{\partial X_y}\right)f(X_x, X_y). \end{aligned} \quad (24)$$

Where $Y_x = \sin(h\partial_t)x(t)$, $X_x = \cos(h\partial_t)x(t)$, $Y_y = \sin(h\partial_t)y(t)$, $X_y = \cos(h\partial_t)y(t)$.

Theorem 7 in reference [1] can be more generally expressed as:

Theorem 2. If $y = f(bx)$ in set of analytic functions is the inverse function of $bx = g(y)$, namely $g(f(bx)) = bx$, then $\sin(h\partial_x)f(bx)$ (denoted by Y) and $\cos(h\partial_x)f(bx)$ (denoted by X) can be determined by the following set of equations:

$$\begin{aligned} \cos\left(Y\frac{\partial}{\partial X}\right)g(X) &= bx, \\ \sin\left(Y\frac{\partial}{\partial X}\right)g(X) &= bh. \end{aligned} \quad x \in \mathbb{R}^n, \quad b \in \mathbb{R}_n, \quad h \in \mathbb{R}_n. \quad (25)$$

By using (20)-(25), we can constructively extend the domains and ranges of the following five abstract operators

$$\exp(h\partial_x), \sin(h\partial_x), \cos(h\partial_x), \sinh(h\partial_x) \text{ and } \cosh(h\partial_x)$$

from power functions and exponential functions to all the C^∞ functions. For instance (See [5])

$$\begin{aligned} \sin(h\partial_x) \arctan bx &= \frac{1}{2} \tanh^{-1} \frac{2bh}{1 + (bx)^2 + (bh)^2}, \\ \cos(h\partial_x) \arctan bx &= \frac{1}{2} \arctan \frac{2bx}{1 - (bx)^2 - (bh)^2}. \end{aligned} \quad (26)$$

Therefore, establishing such a set of algorithms as (20)-(25) is the second important way to extend the domains and ranges of abstract operators.

Because of the introduction of Definition 5, it is necessary to add new algorithms.

Theorem 3. Let $a \in \mathbb{C}^1$, $\rho \in \mathbb{R}^1$, then $\forall f(z) \in C^\infty(\mathbb{C}^1)$, we have

$$a^{\rho \frac{\partial}{\partial \rho}} f(\rho) = f(a\rho). \quad (27)$$

Proof. By Definition 5, for $a \in \mathbb{C}^1$, $\rho \in \mathbb{R}^1$, we have

$$a^{\rho \frac{\partial}{\partial \rho}} \rho^\alpha = a^\alpha \rho^\alpha \quad \text{or} \quad a^{\rho \frac{\partial}{\partial \rho}} \rho^\alpha = (a\rho)^\alpha, \quad \forall \alpha \in \mathbb{N}^1.$$

According to the analytic continuous fundamental theorem, we have Theorem 3.

Theorem 4. Let $X = (\rho_1 \cos \theta_1, \dots, \rho_n \cos \theta_n)$, $Y = (\rho_1 \sin \theta_1, \dots, \rho_n \sin \theta_n)$, introducing the notation

$$\theta \rho \frac{\partial}{\partial \rho} = \theta_1 \rho_1 \frac{\partial}{\partial \rho_1} + \dots + \theta_n \rho_n \frac{\partial}{\partial \rho_n}, \quad Y \partial_X = Y_1 \frac{\partial}{\partial X_1} + \dots + Y_n \frac{\partial}{\partial X_n},$$

then $\forall f(z) \in C^\infty(\mathbb{C}^n)$, $\rho \in \mathbb{R}^n$, $\theta \in \mathbb{R}_n$, we have

$$\begin{aligned} \cos\left(\theta \rho \frac{\partial}{\partial \rho}\right) f(\rho) &= \cos(Y \partial_X) f(X), \\ \sin\left(\theta \rho \frac{\partial}{\partial \rho}\right) f(\rho) &= \sin(Y \partial_X) f(X). \end{aligned} \quad (28)$$

Proof. Let $a_1 = e^{i\theta_1}, \dots, a_n = e^{i\theta_n}$, according to Theorem 3, then

$$\exp\left(i\theta \rho \frac{\partial}{\partial \rho}\right) f(\rho) = f(\rho e^{i\theta}) = f(X + iY) = \exp(iY \partial_X) f(X), \quad X \in \mathbb{R}^n, \quad Y \in \mathbb{R}_n.$$

Considering

$$\begin{aligned}\exp\left(i\theta\rho\frac{\partial}{\partial\rho}\right) &= \cos\left(\theta\rho\frac{\partial}{\partial\rho}\right) + i\sin\left(\theta\rho\frac{\partial}{\partial\rho}\right), \\ \exp(iY\partial_X) &= \cos(Y\partial_X) + i\sin(Y\partial_X),\end{aligned}$$

thus we have Theorem 4, which transforms the abstract operators taking $\rho\frac{\partial}{\partial\rho}$ as the operator element into those taking ∂_X as the operator element.

Corollary 5. Let $f(z)$, $z = x + iy \in \mathbb{C}^1$ be an arbitrary analytic function in a complex plane, then on the line $y = kx$ we have

$$\begin{aligned}\cos\left(y\frac{\partial}{\partial x}\right)f(x)\Big|_{y=kx} &= \beta^{x\frac{\partial}{\partial x}}\cos\left(\alpha x\frac{\partial}{\partial x}\right)f(x), \\ \sin\left(y\frac{\partial}{\partial x}\right)f(x)\Big|_{y=kx} &= \beta^{x\frac{\partial}{\partial x}}\sin\left(\alpha x\frac{\partial}{\partial x}\right)f(x).\end{aligned}\tag{29}$$

Where $\beta = (1 + k^2)^{1/2}$, $\alpha = \arctan k$, which can be used to solve boundary value problems of 2-dimensional Laplace equation on polygonal domains.

Corollary 6. Making use of the analytic continuous fundamental theorem and the Pochhammer symbol $(\lambda)_m$ defined by

$$(\lambda)_m = \frac{\Gamma(\lambda + m)}{\Gamma(\lambda)} = \begin{cases} 1, & (m = 0), \\ \lambda(\lambda + 1) \cdots (\lambda + m - 1) & (m \in \mathbb{N}_+). \end{cases}\tag{30}$$

It is easily seen

$$(-X)_m u(x) = (-1)^m x^m \frac{d^m u}{dx^m}, \quad \forall u(x) \in C^m(\mathbb{R}^1), \quad X = x \frac{d}{dx} = \frac{d}{dz} = D_z.\tag{31}$$

Where $x = e^z$ or $z = \ln x$. Using this set of formulas, we can easily transform the Euler equation into an ordinary differential equation with constant coefficients.

Therefore, for an arbitrary abstract operators, its symbol are C^∞ functions. Further descriptions require concrete analysis under specific circumstances. For instance, the symbol of the abstract operators on the right side of (17) can be further described by using conditions of the Laplace transform. The domain of the abstract operators $P(\partial_x)$, $\sin(h\partial_x)$ and $\cos(h\partial_x)$ are the C^∞ whole space.

2.3 Abstract operators and bounded function

The abstract operators can also acting on the differentiable functions on bounded domain.

For instance, let $a \leq b$, without losing the universality, assuming $a \geq 0$, $b \geq 0$ we have

$$\begin{aligned}na^{n-1} &\leq (a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}) \leq nb^{n-1}. \\ na^{n-1} &\leq \frac{b^n - a^n}{b - a} \leq nb^{n-1} \quad \text{or} \quad a \leq \left(\frac{1}{n} \frac{b^n - a^n}{b - a}\right)^{\frac{1}{n-1}} \leq b.\end{aligned}$$

Therefore, if $a \leq b$, then we have $a \leq c \leq b$, making

$$nc^{n-1} = \frac{b^n - a^n}{b - a} \quad \text{or} \quad \frac{d}{dc}c^n = \frac{b^n - a^n}{b - a}, \quad \forall n \in \mathbb{N}^1.$$

Taking this one as the characteristic equation, according to the analytic continuous fundamental theorem we get the Lagrange mean value theorem, namely

If $a \leq b$, and $f(x) \in C^1[a, b]$, then $\exists c \in [a, b]$ making

$$\frac{d}{dc}f(c) = \frac{f(b) - f(a)}{b - a}, \quad \forall f(x) \in C^1[a, b]. \quad (32)$$

In reference [5], Guangqing Bi and Yuekai Bi discussed the following functions:

$$\begin{aligned} f(x) &= \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S(e^z) \Big|_{z=0}, \\ g(x) &= \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S(e^z) \Big|_{z=0}, \quad z \in \mathbb{R}^1, a < x < b. \end{aligned} \quad (33)$$

Where $f(x), g(x) \in C_0^\infty(\mathbb{R}^1)$, $S(t) \in C^\infty(\Omega_0)$, the Ω_0 is a certain neighborhood at $t = 0$.

If $S(t)$ is expanded in power series on Ω_0 , then (33) are equivalent to Fourier series, and their domain (a, b) can be extended to all $(-\infty, +\infty)$ periodically. So $f(x), g(x) \in L^2([-c, c])$ and $f(x + 2l) = f(x)$, $g(x + 2l) = g(x)$.

For instance, by (23) and (26), we have

$$\begin{aligned} \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) \arctan e^z \Big|_{z=0} &= \cos\left(Y \frac{\partial}{\partial X}\right) \arctan X \Big|_{z=0} \\ &= \frac{1}{2} \arctan \frac{2X}{1 - (X^2 + Y^2)} \Big|_{z=0} = \frac{1}{2} \arctan \frac{2 \cos(\pi x/c)}{1 - (\cos^2(\pi x/c) + \sin^2(\pi x/c))}. \end{aligned}$$

When $\cos(\pi x/c) \neq 0$ or $|x| \neq kc + c/2$, $k = 0, 1, 2, \dots$, the above expression turns into

$$\cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) \arctan e^z \Big|_{z=0} = \begin{cases} (1/2) \arctan(+\infty) = +\pi/4, & 2Kc - c/2 < x < 2Kc + c/2 \\ (1/2) \arctan(-\infty) = -\pi/4, & 2Kc + c/2 < x < 2Kc + 3c/2. \end{cases}$$

Where $K = 0, \pm 1, \pm 2, \dots$. This function represents a square wave.

By algorithms (19)-(25), and some basic formulas similar to (26) (See [5]), we obtained

$$\begin{aligned} \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) (-\ln(1 - e^z)) \Big|_{z=0} &= \frac{\pi}{2} - \frac{\pi x}{2c}, \quad 0 < x < 2c. \\ \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) (-\ln(1 - e^z)) \Big|_{z=0} &= -\ln\left(2 \sin \frac{\pi x}{2c}\right), \quad 0 < x < 2c. \\ \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) \ln(1 + e^z) \Big|_{z=0} &= \frac{\pi x}{2c}, \quad |x| < c. \\ \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) \ln(1 + e^z) \Big|_{z=0} &= \ln\left(2 \cos \frac{\pi x}{2c}\right), \quad |x| < c. \\ \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) \arctan e^z \Big|_{z=0} &= \frac{\pi}{4}, \quad |x| < c/2. \end{aligned} \quad (34)$$

In other words, the ranges of the abstract operators $\sin(h\partial_x)$ and $\cos(h\partial_x)$ may be extended from the $C_0^\infty(\mathbb{R}^1)$ to the $L^2([-c, c])$.

In order to solve an initial-boundary value problems of partial differential equations, we need to express a function with boundary conditions. For instance, (34) are infinitely differentiable within a finite domain, such as

$$\frac{d^k}{dx^k} \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) (-\ln(1 - e^z)) \Big|_{z=0} = \frac{d^k}{dx^k} \left(\frac{\pi}{2} - \frac{\pi x}{2c}\right), \quad \forall k \in \mathbb{N}, 0 < x < 2c.$$

Example 3. $\forall f(z) \in C^\infty(\mathbb{C}^1)$, if $f(z)$ is continuous at $z = 1/2$, then

$$\frac{2}{\pi} \int_0^{\pi/2} \cos\left(x \rho \frac{\partial}{\partial \rho}\right) f(\rho) \Big|_{\rho=\cos x} dx = f\left(\frac{1}{2}\right). \quad (35)$$

Proof. Taking the following integral formula as the characteristic equation

$$\int_0^{\pi/2} \cos^n x \cos nx \, dx = \frac{\pi}{2^{n+1}}, \quad \forall n \in \mathbb{N}.$$

Combining the characteristic equation with the following operator equation by using Definition 5 and the base function x^α :

$$\frac{2}{\pi} \int_0^{\pi/2} \cos\left(x \rho \frac{\partial}{\partial \rho}\right) \rho^n \, dx = \left(\frac{1}{2}\right)^n, \quad \forall n \in \mathbb{N}^1, \quad x \in [0, \pi/2].$$

Where $\rho = \cos x$, according to the analytic continuous fundamental theorem, we get (35).

Example 4. $\forall f(z) \in C^\infty(\mathbb{C}^1)$, if $f(z)$ is differentiable at $z = 1/2$, then

$$\int_0^{\pi/2} \sin\left(x \rho \frac{\partial}{\partial \rho}\right) f(\rho) \Big|_{\rho=\cos x} dx = \frac{1}{2} \int_0^1 \frac{f(\xi) - f(1/2)}{\xi - 1/2} d\xi. \quad (36)$$

Proof. Considering another integral formula

$$\int_0^{\pi/2} \cos^n x \sin nx \, dx = \frac{1}{2^{n+1}} \sum_{k=1}^n \frac{2^k}{k}, \quad \forall n \in \mathbb{N}.$$

Obviously it cannot be taken as the characteristic equation directly, but considering $\sum_{k=1}^n 2^k/k = \int_0^2 \frac{q^n - 1}{q - 1} dq$, and according to the analytic continuous fundamental theorem, we get (36).

Example 5. Let $k \in \mathbb{N}_+$, $\forall f(z) \in C^\infty(\mathbb{C}^1)$, if $f(z)$ is k times continuously differentiable at $z = 1/2$, then

$$\frac{2}{\pi} \int_0^{\pi/2} \cos(2kx) \cos\left(x \rho \frac{\partial}{\partial \rho}\right) f(\rho) \Big|_{\rho=\cos x} dx = \frac{1}{2^{k+1} k!} f^{(k)}\left(\frac{1}{2}\right). \quad (37)$$

$$\frac{2}{\pi} \int_0^{\pi/2} \sin(2kx) \sin\left(x \rho \frac{\partial}{\partial \rho}\right) f(\rho) \Big|_{\rho=\cos x} dx = \frac{1}{2^{k+1} k!} f^{(k)}\left(\frac{1}{2}\right). \quad (38)$$

Proof. In [6], Canon has given the following useful identical equations($\forall n \in \mathbb{N}$):

$$2^n \cos^n x \cos(nx) = \sum_{k=0}^n \binom{n}{k} \cos(2kx). \quad (39)$$

$$2^n \cos^n x \sin(nx) = \sum_{k=0}^n \binom{n}{k} \sin(2kx). \quad (40)$$

Using the orthogonality of trigonometric functions, by (39) and (40) we have

$$\frac{1}{\pi} \int_0^\pi \cos(2kx) \cos^n x \cos(nx) dx = \binom{n}{k} \left(\frac{1}{2}\right)^{n+1}, \quad \forall k \in \mathbb{N}_+. \quad (41)$$

$$\frac{1}{\pi} \int_0^\pi \sin(2kx) \cos^n x \sin(nx) dx = \binom{n}{k} \left(\frac{1}{2}\right)^{n+1}, \quad \forall k \in \mathbb{N}_+. \quad (42)$$

Which can also be written as

$$\frac{2}{\pi} \int_0^{\pi/2} \cos(2kx) \cos^n x \cos(nx) dx = \binom{n}{k} \left(\frac{1}{2}\right)^{n+1}. \quad (43)$$

$$\frac{2}{\pi} \int_0^{\pi/2} \sin(2kx) \cos^n x \sin(nx) dx = \binom{n}{k} \left(\frac{1}{2}\right)^{n+1}. \quad (44)$$

Clearly, (43) is the characteristic equation of (37), and (44) is the characteristic equation of (38).

So by Definition 5 we have

$$\frac{2}{\pi} \int_0^{\pi/2} \cos(2kx) \cos \left(x \rho \frac{\partial}{\partial \rho} \right) \rho^n \Big|_{\rho=\cos x} dx = \frac{1}{2^{k+1} k!} \frac{d^k}{d\xi^k} \xi^n \Big|_{\xi=1/2}, \quad \forall n \in \mathbb{N}^1. \quad (45)$$

$$\frac{2}{\pi} \int_0^{\pi/2} \sin(2kx) \sin \left(x \rho \frac{\partial}{\partial \rho} \right) \rho^n \Big|_{\rho=\cos x} dx = \frac{1}{2^{k+1} k!} \frac{d^k}{d\xi^k} \xi^n \Big|_{\xi=1/2}, \quad \forall n \in \mathbb{N}^1. \quad (46)$$

Then according to the analytic continuous fundamental theorem, we get (37) and (38).

In reference [5] we get

Let $S(t) = \sum_{n=0}^\infty a_n t^n$, $t \in \mathbb{R}^1$, $0 \leq t \leq r$, $0 < r < +\infty$, if

$$\begin{aligned} \sum_{n=0}^\infty a_n \cos \frac{n\pi x}{c} &= \cos \left(\frac{\pi x}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0}, \\ \sum_{n=0}^\infty a_n \sin \frac{n\pi x}{c} &= \sin \left(\frac{\pi x}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0} \end{aligned}$$

are tenable in $a < x < b$, $x \in \mathbb{R}^1$, then $\forall x_0 \in [0, \frac{b-a}{2})$,

$$\begin{aligned} \sum_{n=0}^\infty a_n \cos \frac{n\pi x_0}{c} \cos \frac{n\pi x}{c} &= \cosh \left(x_0 \frac{\partial}{\partial x} \right) \cos \left(\frac{\pi x}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0}, \\ \sum_{n=0}^\infty a_n \cos \frac{n\pi x_0}{c} \sin \frac{n\pi x}{c} &= \cosh \left(x_0 \frac{\partial}{\partial x} \right) \sin \left(\frac{\pi x}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0} \end{aligned} \quad (47)$$

are tenable in $a + x_0 < x < b - x_0$. Accordingly, for any definite value $x \in (a, b)$, x_0 in (47) takes values in the following interval:

$$0 < x_0 < \begin{cases} x - a, & a < x \leq (a+b)/2, \\ b - x, & b > x \geq (a+b)/2. \end{cases}$$

Where the left side of (47) is:

$$\begin{aligned}\cos \frac{n\pi x_0}{c} \cos \frac{n\pi x}{c} &= \cosh \left(x_0 \frac{\partial}{\partial x} \right) \cos \frac{n\pi x}{c}, \\ \cos \frac{n\pi x_0}{c} \sin \frac{n\pi x}{c} &= \cosh \left(x_0 \frac{\partial}{\partial x} \right) \sin \frac{n\pi x}{c}.\end{aligned}$$

Therefore, $\sinh(h\partial_x)$ and $\cosh(h\partial_x)$ are the continuous operators on $L^2([-c, c])$.

2.4 Abstract operators and Fourier analysis

The third way to extend the domains and ranges of abstract operators is using Fourier analysis.

Theorem 5. Let $t \in \overline{\mathbb{R}_+^1}$, $\forall f(x) \in S(\mathbb{R}^1)$, if $F_+(s)$, $F_-(s)$ are respectively

$$F_+(s) = \frac{1}{\pi} \mathcal{L} \int_{-\infty}^{\infty} f(\xi) \cos(t\xi) d\xi, \quad F_-(s) = \frac{1}{\pi} \mathcal{L} \int_{-\infty}^{\infty} f(\xi) \sin(t\xi) d\xi, \quad (48)$$

where \mathcal{L} is the Laplace transform operator, then we have

$$f(x) = \cos \left(x \frac{\partial}{\partial s} \right) F_+(s) \Big|_{s=0} + \sin \left(-x \frac{\partial}{\partial s} \right) F_-(s) \Big|_{s=0}. \quad (49)$$

Or

$$\cos \left(x \frac{\partial}{\partial s} \right) F_+(s) \Big|_{s=0} = \frac{f(x) + f(-x)}{2}, \quad \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(tx) dx = \mathcal{L}^{-1} F_+(s). \quad (50)$$

$$\sin \left(-x \frac{\partial}{\partial s} \right) F_-(s) \Big|_{s=0} = \frac{f(x) - f(-x)}{2}, \quad \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(tx) dx = \mathcal{L}^{-1} F_-(s). \quad (51)$$

Proof. By (17) we have

$$\mathcal{L} \cos(t\xi) = \cos \left(\xi \frac{\partial}{\partial s} \right) \frac{1}{s}, \quad \mathcal{L} \sin(t\xi) = \sin \left(-\xi \frac{\partial}{\partial s} \right) \frac{1}{s}, \quad t \in \overline{\mathbb{R}_+^1}, \quad \Re(s) > 0.$$

So by (48) and (49), $\forall f(x) \in S(\mathbb{R}^1)$, $\exists f_s(x) \in C^\infty(\mathbb{R}^1)$ can be expressed as

$$\begin{aligned}f_s(x) &= \cos \left(x \frac{\partial}{\partial s} \right) F_+(s) + \sin \left(-x \frac{\partial}{\partial s} \right) F_-(s) \\ &= \cos \left(x \frac{\partial}{\partial s} \right) \left[\frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \mathcal{L} \cos(t\xi) d\xi \right] + \sin \left(-x \frac{\partial}{\partial s} \right) \left[\frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \mathcal{L} \sin(t\xi) d\xi \right] \\ &= \frac{1}{\pi} \cos \left(x \frac{\partial}{\partial s} \right) \int_{-\infty}^{\infty} f(\xi) \cos \left(\xi \frac{\partial}{\partial s} \right) \frac{1}{s} d\xi + \frac{1}{\pi} \sin \left(-x \frac{\partial}{\partial s} \right) \int_{-\infty}^{\infty} f(\xi) \sin \left(-\xi \frac{\partial}{\partial s} \right) \frac{1}{s} d\xi \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \left[\cos \left(x \frac{\partial}{\partial s} \right) \cos \left(\xi \frac{\partial}{\partial s} \right) + \sin \left(-x \frac{\partial}{\partial s} \right) \sin \left(-\xi \frac{\partial}{\partial s} \right) \right] \frac{1}{s} d\xi \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \cos \left((x - \xi) \frac{\partial}{\partial s} \right) \frac{1}{s} d\xi = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \frac{s}{(x - \xi)^2 + s^2} d\xi, \quad \Re(s) > 0.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\lim_{s \rightarrow 0^+} f_s(x) &= \lim_{s \rightarrow 0^+} \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \frac{s}{(x - \xi)^2 + s^2} d\xi \\ &= \int_{-\infty}^{\infty} f(\xi) \lim_{s \rightarrow 0^+} \frac{1}{\pi} \frac{s}{(x - \xi)^2 + s^2} d\xi \rightarrow \int_{-\infty}^{\infty} f(\xi) \delta(x - \xi) d\xi = f(x).\end{aligned}$$

In the same manner we find

Theorem 6. Let $f(x) \in L^2([-l, l])$, $f(x + 2l) = f(x)$. If $S_+(t)$ and $S_-(t)$ are respectively

$$S_+(t) = \frac{1}{2l} \int_{-l}^l f(\xi) \frac{1 - t^2}{1 - 2t \cos(\pi \xi / l) + t^2} d\xi, \quad (52)$$

$$S_-(t) = \frac{1}{l} \int_{-l}^l f(\xi) \frac{t \sin(\pi \xi / l)}{1 - 2t \cos(\pi \xi / l) + t^2} d\xi, \quad (53)$$

then $\exists f_z(x) \in C^\infty(\mathbb{R}^1)$ with the following form

$$f_z(x) = \cos\left(\frac{\pi x}{l} \frac{\partial}{\partial z}\right) S_+(e^z) + \sin\left(\frac{\pi x}{l} \frac{\partial}{\partial z}\right) S_-(e^z), \quad -\infty < z < 0 \quad (54)$$

making

$$\begin{aligned} \lim_{z \rightarrow 0^-} f_z(x) &\rightarrow f(x), \quad \forall f(x) \in C(-l, l) \subset L^2([-l, l]). \\ \lim_{z \rightarrow 0^-} \|f(x) - f_z(x)\|_{L^2([-l, l])} &= 0, \quad \forall f(x) \in L^2([-l, l]). \end{aligned}$$

Proof. By the algorithms (21) we have

$$\begin{aligned} \cos\left(\frac{\pi \xi}{l} \frac{\partial}{\partial z}\right) \frac{1 + e^z}{1 - e^z} &= \frac{1 - e^{2z}}{1 - 2e^z \cos(\pi \xi / l) + e^{2z}}. \\ \sin\left(\frac{\pi \xi}{l} \frac{\partial}{\partial z}\right) \frac{e^z}{1 - e^z} &= \frac{e^z \sin(\pi \xi / l)}{1 - 2e^z \cos(\pi \xi / l) + e^{2z}}. \end{aligned}$$

So $f_z(x)$ can be expressed as

$$\begin{aligned} f_z(x) &= \cos\left(\frac{\pi x}{l} \frac{\partial}{\partial z}\right) \left[\frac{1}{2l} \int_{-l}^l f(\xi) \cos\left(\frac{\pi \xi}{l} \frac{\partial}{\partial z}\right) \frac{1 + e^z}{1 - e^z} d\xi \right] \\ &\quad + \sin\left(\frac{\pi x}{l} \frac{\partial}{\partial z}\right) \left[\frac{1}{l} \int_{-l}^l f(\xi) \sin\left(\frac{\pi \xi}{l} \frac{\partial}{\partial z}\right) \frac{e^z}{1 - e^z} d\xi \right] \\ &= \frac{1}{2l} \int_{-l}^l f(\xi) \cos\left(\frac{\pi x}{l} \frac{\partial}{\partial z}\right) \cos\left(\frac{\pi \xi}{l} \frac{\partial}{\partial z}\right) \left[1 + \frac{2e^z}{1 - e^z} \right] d\xi \\ &\quad + \frac{1}{l} \int_{-l}^l f(\xi) \sin\left(\frac{\pi x}{l} \frac{\partial}{\partial z}\right) \sin\left(\frac{\pi \xi}{l} \frac{\partial}{\partial z}\right) \frac{e^z}{1 - e^z} d\xi = \frac{1}{2l} \int_{-l}^l f(\xi) d\xi \\ &\quad + \frac{1}{l} \int_{-l}^l f(\xi) \left[\cos\left(\frac{\pi x}{l} \frac{\partial}{\partial z}\right) \cos\left(\frac{\pi \xi}{l} \frac{\partial}{\partial z}\right) + \sin\left(\frac{\pi x}{l} \frac{\partial}{\partial z}\right) \sin\left(\frac{\pi \xi}{l} \frac{\partial}{\partial z}\right) \right] \frac{e^z}{1 - e^z} d\xi \\ &= \frac{1}{2l} \int_{-l}^l f(\xi) d\xi + \frac{1}{l} \int_{-l}^l f(\xi) \cos\left(\frac{\pi(x - \xi)}{l} \frac{\partial}{\partial z}\right) \frac{e^z}{1 - e^z} d\xi \\ &= \frac{1}{2l} \int_{-l}^l f(\xi) \cos\left(\frac{\pi(x - \xi)}{l} \frac{\partial}{\partial z}\right) \frac{1 + e^z}{1 - e^z} d\xi \\ &= \frac{1}{2l} \int_{-l}^l f(\xi) \frac{1 - e^{2z}}{1 - 2e^z \cos(\pi(x - \xi)/l) + e^{2z}} d\xi, \quad -\infty < z < 0. \end{aligned}$$

Therefore, if $f(x) \in C(-l, l) \subset L^2([-l, l])$, we get

$$\begin{aligned}\lim_{z \rightarrow 0^-} f_z(x) &= \lim_{z \rightarrow 0^-} \frac{1}{2l} \int_{-l}^l f(\xi) \frac{1 - e^{2z}}{1 - 2e^z \cos(\pi(x - \xi)/l) + e^{2z}} d\xi \\ &= \int_{-l}^l f(\xi) \lim_{z \rightarrow 0^-} \frac{1}{2l} \frac{1 - e^{2z}}{1 - 2e^z \cos(\pi(x - \xi)/l) + e^{2z}} d\xi \rightarrow \int_{-l}^l f(\xi) \delta(x - \xi) d\xi = f(x).\end{aligned}$$

As we all know, $\forall f(x) \in L^2([-l, l])$, $f(x + 2l) = f(x)$ we have

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi x}{l} + b_k \sin \frac{k\pi x}{l} \right).$$

Where $\forall k \in \mathbb{N}_+$

$$a_0 = \frac{1}{2l} \int_{-l}^l f(\xi) d\xi, \quad a_k = \frac{1}{l} \int_{-l}^l f(\xi) \cos \frac{k\pi x}{l} d\xi, \quad b_k = \frac{1}{l} \int_{-l}^l f(\xi) \sin \frac{k\pi x}{l} d\xi.$$

Accordingly $f_z(x) \in C^\infty(\mathbb{R}^1)$ can be expressed as

$$f_z(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k e^{kz} \cos \frac{k\pi x}{l} + b_k e^{kz} \sin \frac{k\pi x}{l} \right), \quad -\infty < z < 0. \quad (55)$$

So we have

$$\frac{1}{l} \int_{-l}^l |f(x) - f_z(x)|^2 dx = \sum_{k=1}^{\infty} [(a_k^2 + b_k^2)(1 - e^{2kz})], \quad -\infty < z < 0. \quad (56)$$

$$\lim_{z \rightarrow 0^-} \|f(x) - f_z(x)\|_{L^2([-l, l])} = 0, \quad \forall f(x) \in L^2([-l, l]), \quad f(x + 2l) = f(x). \quad (57)$$

If $f(x)$ is an even function, then we have the integral equation:

$$\frac{1}{l} \int_0^l f(x) \frac{1 - t^2}{1 - 2t \cos(\pi x/l) + t^2} dx = S_+(t), \quad \forall S_+(t) \in C^\infty(\Omega_0). \quad (58)$$

Its solution $f(x) \in L^2([-l, l])$, $f(x + 2l) = f(x)$, and we have

$$f(x) = \cos \left(\frac{\pi x}{l} \frac{\partial}{\partial z} \right) S_+(e^z) \Big|_{z=0}. \quad (59)$$

If $f(x)$ is an odd function, then we have the integral equation:

$$\frac{2}{l} \int_0^l f(x) \frac{t \sin(\pi x/l)}{1 - 2t \cos(\pi x/l) + t^2} dx = S_-(t), \quad \forall S_-(t) \in C^\infty(\Omega_0). \quad (60)$$

Its solution $f(x) \in L^2([-l, l])$, $f(x + 2l) = f(x)$, and we have

$$f(x) = \sin \left(\frac{\pi x}{l} \frac{\partial}{\partial z} \right) S_-(e^z) \Big|_{z=0}. \quad (61)$$

For instance, in the integral equation (58), if $S_+(t) = \arctan t$, $0 < t < 1$, then its solution is a square wave, namely

$$f(x) = \begin{cases} +\pi/4, & 2Kl - l/2 < x < 2Kl + l/2 \\ -\pi/4, & 2Kl + l/2 < x < 2Kl + 3l/2. \end{cases}$$

Where $K = 0, \pm 1, \pm 2, \dots$.

Let $\rho_z(x) \in C_0^\infty(\mathbb{R}^n)$, $\text{supp}\rho_z(x) \subset \{x \in \mathbb{R}^n \mid |x| \leq l\}$ be a polished kernel, taken as $\rho_z(x) = \rho_{z_1}(x_1)\rho_{z_2}(x_2)\cdots\rho_{z_i}(x_i)\cdots\rho_{z_n}(x_n)$ and

$$\rho_{z_i}(x_i) = \frac{1}{2l_i} \frac{1 - e^{2z_i}}{1 - 2e^{z_i} \cos(\pi x_i/l_i) + e^{2z_i}}, \quad -\infty < z_i < 0, \quad |x_i| \leq l_i \quad (i = 1, 2, \dots, n). \quad (62)$$

Then we easily prove the following results:

$$\lim_{z \rightarrow 0^-} g(\partial_x) \int_{-l}^l f(\xi) \rho_z(x - \xi) d\xi \rightarrow g(\partial_x) f(x), \quad \forall f(x) \in L^2([-l, l]), \quad g(\xi) \in C^\infty(\mathbb{R}_n). \quad (63)$$

Where $f(x + 2l) = f(x)$, $l \in \mathbb{R}^n$, $[-l, l] = \{x \in \mathbb{R}^n \mid -l \leq x \leq l\}$.

If $g(\partial_x) = \partial_x^\alpha$, $\forall \alpha \in \mathbb{N}^n$, then (63) gives a definition of generalized derivative on $L^2([-l, l])$.

2.5 Integral representation of abstract operators

Clearly, such algorithms as (20)-(25) do not exist for most abstract operators. Therefore, for more complex abstract operators, establishing the integral expression is the fourth important way to extend the domain of abstract operators. In reference [2] Guangqing Bi has obtained the following results:

Theorem BI1. Let $P(\partial_x)$ be an m -order partial differential operator of any kind and there exist a_1, a_2, \dots, a_k of real and partial differential operators A_1, A_2, \dots, A_k of the order less than $[(m+1)/2]$ such that $P(\partial_x) \equiv a_1 A_1^2 + a_2 A_2^2 + \dots + a_k A_k^2$. If $k = 2\nu + 3$, $\nu = 0, 1, 2, \dots$, then $\forall f(x) \in C^{m\nu}(\mathbb{R}^n)$ we have

$$\begin{aligned} \frac{\sinh(\alpha t P(\partial_x)^{1/2})}{\alpha P(\partial_x)^{1/2}} f(x) &= \underbrace{t \int_0^t t dt \cdots \int_0^t t dt}_{\nu} \frac{(\alpha^2 P(\partial_x))^\nu}{2^{\nu+2} \pi^{\nu+1}} \\ &\times \underbrace{\int_{-\pi}^\pi \int_0^\pi \cdots \int_0^\pi}_{k-2} e^{\eta_1 a_1^{1/2} A_1 + \cdots + \eta_k a_k^{1/2} A_k} f(x) d\sigma_k \\ &+ \sum_{i=0}^{\nu-1} \frac{t^{2i+1}}{(2i+1)!} (\alpha^2 P(\partial_x))^i f(x). \end{aligned} \quad (64)$$

Where α is a real numbers, $x \in \mathbb{R}^n$, $t \in \mathbb{R}^1$. $\eta \in \mathbb{R}_k$ is the integral variable and

$$\begin{aligned} \eta_1 &= \alpha t \cos \theta_1, \\ \eta_2 &= \alpha t \sin \theta_1 \cos \theta_2, \\ &\dots \\ \eta_p &= \alpha t \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{p-1} \cos \theta_p, \\ \eta_{p+1} &= \alpha t \sin \theta_1 \sin \theta_2 \cdots \sin \theta_p \cos \phi, \\ \eta_{p+2} &= \eta_k = \alpha t \sin \theta_1 \sin \theta_2 \cdots \sin \theta_p \sin \phi; \end{aligned}$$

$$d\sigma_k = \sin^{k-2} \theta_1 \sin^{k-3} \theta_2 \cdots \sin \theta_{k-2} d\theta_1 d\theta_2 \cdots d\theta_{k-2} d\phi.$$

When A_1, \dots, A_k in the right-hand of (64) are one order partial differential operators, then the abstract operator $e^{\eta_1 a_1^{1/2} A_1 + \cdots + \eta_k a_k^{1/2} A_k}$ is one of the following five simplest operators:

$$\exp(h\partial_x), \sin(h\partial_x), \cos(h\partial_x), \sinh(h\partial_x) \text{ and } \cosh(h\partial_x).$$

Proof. In (64), let $f(x) = e^{\xi x}$, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}_n$, and the symbol of the partial differential operator $A_j, j = 1, 2, \dots, k$ is denoted by $\chi_j(\xi)$, $\beta_j = a_j^{1/2} \chi_j(\xi)$. By Definition 4, the formula degenerates to its characteristic equation:

$$\begin{aligned} \frac{\sinh(\alpha t P(\xi)^{1/2})}{\alpha P(\xi)^{1/2}} &= t \underbrace{\int_0^t t dt \cdots \int_0^t t dt}_{\nu} \frac{(\alpha^2 P(\xi))^\nu}{2^{\nu+2} \pi^{\nu+1}} \int_{-\pi}^\pi \underbrace{\int_0^\pi \cdots \int_0^\pi}_{k-2} e^{\eta_1 \beta_1 + \cdots + \eta_k \beta_k} d\sigma_k \\ &+ \sum_{i=0}^{\nu-1} \frac{t^{2i+1}}{(2i+1)!} (\alpha^2 P(\xi))^i, \quad \nu = \frac{k-3}{2}. \end{aligned} \quad (65)$$

According to the analytic continuous fundamental theorem, we only need to prove (65). Solving the integral on a hypersphere on the right side of (65), we have

$$\frac{\sinh(\alpha t P(\xi)^{1/2})}{\alpha P(\xi)^{1/2}} = t \underbrace{\int_0^t t dt \cdots \int_0^t t dt}_{\nu} \sum_{j=0}^{\infty} \frac{(\alpha^2 P(\xi))^{\nu+j} t^{2j}}{(2j)!! (2j+2\nu+1)!!} + \sum_{i=0}^{\nu-1} \frac{t^{2i+1} (\alpha^2 P(\xi))^i}{(2i+1)!}.$$

Then it is proved by the termwise integration of the infinite series on the right side of the equation.

Similarly, we have (See [1])

Theorem BI1'. Let $P(\partial_x)$ be an m -order partial differential operator of any kind and there exist a_1, a_2, \dots, a_k of real and partial differential operators A_1, A_2, \dots, A_k of the order less than $[(m+1)/2]$ such that $P(\partial_x) \equiv a_1 A_1^2 + a_2 A_2^2 + \cdots + a_k A_k^2$. If $k = 2\nu + 3$, $\nu = 0, 1, 2, \dots$, then $\forall f(z) \in C^\infty(\mathbb{C}^n)$

$$\begin{aligned} \frac{\sin(t P(\partial_x)^{1/2})}{P(\partial_x)^{1/2}} f(x) &= t \underbrace{\int_0^t t dt \cdots \int_0^t t dt}_{\nu} [-P(\partial_x)]^\nu \\ &\times \left(\frac{2}{\pi}\right)^{\nu+1} \underbrace{\int_0^{\pi/2} \cdots \int_0^{\pi/2}}_{k-1} \cos(\eta_1 a_1^{1/2} A_1) \cdots \cos(\eta_k a_k^{1/2} A_k) f(x) d\sigma_k \\ &+ \sum_{i=0}^{\nu-1} \frac{t^{2i+1}}{(2i+1)!} [-P(\partial_x)]^i f(x). \end{aligned} \quad (66)$$

Where $x \in \mathbb{R}^n$, $t \in \mathbb{R}^1$. $\eta \in \mathbb{R}_k$ is the integral variable and

$$\begin{aligned} \eta_1 &= t \cos \theta_1, \\ \eta_2 &= t \sin \theta_1 \cos \theta_2, \end{aligned}$$

$$\begin{aligned}
& \dots \\
\eta_p &= t \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{p-1} \cos \theta_p, \\
\eta_{p+1} &= t \sin \theta_1 \sin \theta_2 \cdots \sin \theta_p \cos \phi, \\
\eta_{p+2} &= \eta_k = t \sin \theta_1 \sin \theta_2 \cdots \sin \theta_p \sin \phi;
\end{aligned}$$

$$d\sigma_k = \sin^{k-2} \theta_1 \sin^{k-3} \theta_2 \cdots \sin \theta_{k-2} d\theta_1 d\theta_2 \cdots d\theta_{k-2} d\phi.$$

If A_1, \dots, A_k in (64) and (66) are partial differential operators of the order great than 1, then the order can be lowered by taking the following Theorem:

Theorem BI2. Let $P(\partial_x)$, $x \in \mathbb{R}^n$ be a partial differential operator of any order and $f(x) \in C(\mathbb{R}^n)$ which make the integral in the follow formula meaningful, then

$$e^{\lambda P(\partial_x)} f(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\zeta^2/4} e^{\lambda^{1/2} \zeta P(\partial_x)^{1/2}} f(x) d\zeta, \quad \forall \lambda \in \mathbb{C}^1. \quad (67)$$

Continuing the processing one can arrive at order one.

Example 6. By using the Theorem BI2 and Corollary 2 we have

$$\begin{aligned}
\exp \left(-a^2 t \frac{\partial^2}{\partial x_j^2} \right) g(x) &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\zeta^2/4} \cos \left(a\sqrt{t} \zeta \frac{\partial}{\partial x_j} \right) g(x) d\zeta, \quad t \in \overline{\mathbb{R}_+^1}. \\
\cos \left(\lambda \frac{\partial^2}{\partial x_j^2} \right) g(x) &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\zeta^2/4} e^{\sqrt{\lambda/2} \zeta \frac{\partial}{\partial x_j}} \cos \left(\sqrt{\frac{\lambda}{2}} \zeta \frac{\partial}{\partial x_j} \right) g(x) d\zeta, \\
\sin \left(a\lambda \frac{\partial}{\partial x_j} + \lambda \frac{\partial^2}{\partial x_j^2} \right) g(x) &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\zeta^2/4} e^{\sqrt{\lambda/2} \zeta \frac{\partial}{\partial x_j}} \sin \left(h_{\lambda,a}(\zeta) \frac{\partial}{\partial x_j} \right) g(x) d\zeta.
\end{aligned}$$

Where $h_{\lambda,a}(\zeta) = a\lambda + \sqrt{\lambda/2} \zeta$, $\forall \lambda, a \in \mathbb{C}^1$.

With the expansion of definitions of abstract operators, it is necessary to add new algorithms continuously.

3 Abstract operators and linear higher-order partial differential equations

3.1 Applications of Laplace transform for n-dimensional space

Solving the ordinary or partial differential equations, is constructing the algorithms of the inverse operators of ordinary or partial differential operators. In terms of abstract operators, solving the initial value problem of partial differential equations for n-dimensional space is similar to solving the ordinary differential equations with respect to the variable t , thus we can introduce the Laplace transform to further simplify the solving process.

Corollary 7. Let $g(\int_0^t \cdot t dt)$ be the abstract operators taking $\int_0^t \cdot t dt$ as the operator element, and $g(-\frac{1}{s} \frac{\partial}{\partial s})$ be the abstract operator taking $-\frac{1}{s} \frac{\partial}{\partial s}$ as the operator element, and the symbols of

these two abstract operators are the same, then for any Laplace transformable function $f(t)$ we have

$$g\left(\int_0^t \cdot t dt\right) f(t) = \mathcal{L}^{-1} g\left(-\frac{1}{s} \frac{\partial}{\partial s}\right) \mathcal{L} f(t), \quad t \in \overline{\mathbb{R}_+^1}. \quad (68)$$

Where \mathcal{L} and \mathcal{L}^{-1} are the Laplace transform operator and its inverse operator respectively, and $\mathcal{L} f(t) = F(s)$.

Proof. According to the properties of Laplace transform, we generally have

$$\left(\int_0^t \cdot t dt\right)^m f(t) = \mathcal{L}^{-1} \left(-\frac{1}{s} \frac{\partial}{\partial s}\right)^m F(s), \quad \forall m \in \mathbb{N}.$$

According to the abstract operator fundamental theorem, we get the Corollary 7.

Similarly we have

$$g\left(\int_0^t \cdot dt\right) f(t) = \mathcal{L}^{-1} g\left(\frac{1}{s}\right) \mathcal{L} f(t), \quad t \in \overline{\mathbb{R}_+^1}. \quad (69)$$

Therefore, using properties of the Laplace transform and the abstract operator fundamental theorem, we respectively obtain the abstract operators taking $\int_0^t \cdot t dt$ and $\int_0^t \cdot dt$ as the operator elements. This type of formulas can also be applied in solving certain integral equations.

Theorem 7. Let $m \in \mathbb{N}_+$, $P(\partial_x)$ be arbitrary order partial differential equations for n -dimensional space. Then we have

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - P(\partial_x)\right)^m u = f(x, t), & x \in \mathbb{R}^n, t \in \overline{\mathbb{R}_+^1}, \forall m \in \mathbb{N}_+, \\ \frac{\partial^j u}{\partial t^j} \Big|_{t=0} = \varphi_j(x), & j = 0, 1, 2, \dots, 2m-1. \end{cases} \quad (70)$$

$$\begin{aligned} u(x, t) &= \int_0^t \int_0^{t-\tau} \frac{[(t-\tau)^2 - \tau'^2]^{m-2}}{(2m-2)!!(2m-4)!!} \frac{\sinh(\tau' P(\partial_x)^{1/2})}{P(\partial_x)^{1/2}} f(x, \tau) \tau' d\tau' d\tau \\ &+ \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} P(\partial_x)^k \sum_{j=0}^{2m-1-2k} \frac{\partial^{2m-1-2k-j}}{\partial t^{2m-1-2k-j}} \int_0^t \frac{(t^2 - \tau^2)^{m-2} \tau}{(2m-2)!!(2m-4)!!} \\ &\times \frac{\sinh(\tau P(\partial_x)^{1/2})}{P(\partial_x)^{1/2}} \varphi_j(x) d\tau. \end{aligned} \quad (71)$$

Proof. The Laplace transform of the partial differential equation, with respect to t and considering the initial condition, is

$$\sum_{k=0}^m (-1)^k \binom{m}{k} P(\partial_x)^k \left(s^{2m-2k} U(x, s) - \sum_{j=0}^{2m-1-2k} s^{2m-1-2k-j} \varphi_j(x) \right) = F(x, s).$$

Where $U(x, s) = \mathcal{L} u(x, t)$, $F(x, s) = \mathcal{L} f(x, t)$. Let $G_m(\partial_x, t) = \mathcal{L}^{-1}[1/(s^2 - P(\partial_x))^m]$, solving $U(x, s)$ and its inverse Laplace transform is

$$u(x, t) = \mathcal{L}^{-1} U(x, s) = \mathcal{L}^{-1} \frac{1}{(s^2 - P(\partial_x))^m} F(x, s)$$

$$\begin{aligned}
& + \mathcal{L}^{-1} \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} P(\partial_x)^k \sum_{j=0}^{2m-1-2k} \frac{s^{2m-1-2k-j}}{(s^2 - P(\partial_x))^m} \varphi_j(x) \\
& = G_m(\partial_x, t) * f(x, t) \\
& + \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} P(\partial_x)^k \sum_{j=0}^{2m-1-2k} \frac{\partial^{2m-1-2k-j}}{\partial t^{2m-1-2k-j}} G_m(\partial_x, t) \varphi_j(x).
\end{aligned}$$

Now let us solve $G_m(\partial_x, t)$. In Corollary 7, taking $g(\xi) = \xi^{m-1}$, $\xi \in \mathbb{R}_1$ as the symbol of the abstract operator, and let $f(t) = \sin bt$, $t \in \mathbb{R}^1$, we have

$$\left(\int_0^t \cdot t dt \right)^{m-1} \sin bt = \mathcal{L}^{-1} \left(-\frac{1}{s} \frac{\partial}{\partial s} \right)^{m-1} \frac{b}{s^2 + b^2} = \mathcal{L}^{-1} \frac{2^{m-1}(m-1)!}{(s^2 + b^2)^m} b.$$

Let $b = iP(\xi)^{1/2}$, $\xi \in \mathbb{R}_n$, we have

$$\mathcal{L}^{-1} \frac{1}{(s^2 - P(\xi))^m} = \frac{1}{(2m-2)!!} \left(\int_0^t \cdot t dt \right)^{m-1} \frac{\sinh(tP(\xi)^{1/2})}{P(\xi)^{1/2}}.$$

Taking this one as the characteristic equation, according to the analytic continuous fundamental theorem, we have

$$G_m(\partial_x, t) = \mathcal{L}^{-1} \frac{1}{(s^2 - P(\partial_x))^m} = \frac{1}{(2m-2)!!} \left(\int_0^t \cdot t dt \right)^{m-1} \frac{\sinh(tP(\partial_x)^{1/2})}{P(\partial_x)^{1/2}}. \quad (72)$$

By using (8) in reference [3], we can easily derive the following integral formula

$$\left(\int_a^x \cdot x dx \right)^m f(x) = \underbrace{\int_a^x x dx \cdots \int_a^x x dx}_m = \int_a^x \frac{(x^2 - \xi^2)^{m-1}}{(2m-2)!!} f(\xi) \xi d\xi. \quad (73)$$

Applying (73) to (72), we have the expression of abstract operator $G_m(\partial_x, t)$:

$$G_m(\partial_x, t)g(x) = \int_0^t \frac{(t^2 - \tau'^2)^{m-2}}{(2m-2)!!(2m-4)!!} \frac{\sinh(\tau'P(\partial_x)^{1/2})}{P(\partial_x)^{1/2}} g(x) \tau' d\tau'. \quad (74)$$

Thus Theorem 7 is proved.

In reference [2] the Guangqing Bi has obtained the following results:

Theorem BI3. Let a_1, a_2, \dots, a_m be arbitrary real or complex numbers different from each other, $P(\partial_x)$ be a partial differential operator of any order, then we have

$$\begin{cases} \prod_{i=1}^m \left(\frac{\partial}{\partial t} - a_i P(\partial_x) \right) u = f(x, t), & x \in \mathbb{R}^n, t \in \overline{\mathbb{R}_+^1}, \forall m \in \mathbb{N}_+, \\ \left. \frac{\partial^j u}{\partial t^j} \right|_{t=0} = 0, & j = 0, 1, 2, \dots, m-1. \end{cases} \quad (75)$$

$$u(x, t) = \int_0^t \int_0^{t-\tau} \frac{(t - \tau - \tau')^{m-2}}{(m-2)!} \sum_{j=1}^m \frac{a_j^{m-1}}{\prod_{i \neq j}^m (a_j - a_i)} e^{\tau' a_j P(\partial_x)} f(x, \tau) d\tau' d\tau. \quad (76)$$

Theorem BI4. Let a_1, a_2, \dots, a_m be arbitrary real or complex numbers different from each other, $P(\partial_x)$ be a partial differential operator of any order, then we have

$$\begin{cases} \prod_{i=1}^m (\frac{\partial^2}{\partial t^2} - a_i^2 P(\partial_x))u = f(x, t), & x \in \mathbb{R}^n, t \in \overline{\mathbb{R}_+^1}, \forall m \in \mathbb{N}_+, \\ \frac{\partial^j u}{\partial t^j} \Big|_{t=0} = 0, & j = 0, 1, 2, \dots, 2m-1. \end{cases} \quad (77)$$

$$u(x, t) = \int_0^t \int_0^{t-\tau} \frac{(t-\tau-\tau')^{2m-3}}{(2m-3)!} \sum_{j=1}^m \frac{a_j^{2m-2}}{\prod_{i \neq j}^m (a_j^2 - a_i^2)} \frac{\sinh(\tau' a_j P(\partial_x)^{1/2})}{a_j P(\partial_x)^{1/2}} f(x, \tau) d\tau' d\tau. \quad (78)$$

On this basis, by using the abstract operators and Laplace transform we have obtained the following results:

Theorem 8. Let a_1, a_2, \dots, a_m be arbitrary real or complex roots different from each other for $b_0 + b_1\chi + b_2\chi^2 + \dots + b_m\chi^m = 0$, and $P(\partial_x, \partial_t)$ be a partial differential operator defined by

$$P(\partial_x, \partial_t) = \sum_{k=0}^m b_k P(\partial_x)^{m-k} \frac{\partial^k}{\partial t^k}, \quad x \in \mathbb{R}^n, t \in \overline{\mathbb{R}_+^1}, \forall m \in \mathbb{N}_+.$$

Where $P(\partial_x)$ is a partial differential operator of any order. Then we have

$$\begin{cases} P(\partial_x, \partial_t)u = f(x, t), & x \in \mathbb{R}^n, t \in \overline{\mathbb{R}_+^1}, \forall m \in \mathbb{N}_+, \\ \frac{\partial^r u}{\partial t^r} \Big|_{t=0} = \varphi_r(x), & r = 0, 1, 2, \dots, m-1. \end{cases} \quad (79)$$

$$\begin{aligned} u(x, t) &= \int_0^t \int_0^{t-\tau} \frac{(t-\tau-\tau')^{m-2}}{(m-2)!} \sum_{j=1}^m \frac{a_j^{m-1}}{\prod_{i \neq j}^m (a_j - a_i)} e^{\tau' a_j P(\partial_x)} f(x, \tau) d\tau' d\tau \\ &+ \sum_{k=1}^m b_k P(\partial_x)^{m-k} \sum_{r=0}^{k-1} \frac{\partial^{k-1-r}}{\partial t^{k-1-r}} \int_0^t \frac{(t-\tau)^{m-2}}{(m-2)!} \\ &\times \sum_{j=1}^m \frac{a_j^{m-1}}{\prod_{i \neq j}^m (a_j - a_i)} e^{\tau a_j P(\partial_x)} \varphi_r(x) d\tau. \end{aligned} \quad (80)$$

Theorem 9. Let a_1, a_2, \dots, a_m be arbitrary real or complex roots different from each other, satisfy $\sum_{k=0}^m b_{2k}\chi^{2k} = \prod_{i=1}^m (\chi^2 - a_i^2)$, and $P(\partial_x, \partial_t)$ be a partial differential operators defined by

$$P(\partial_x, \partial_t) = \sum_{k=0}^m b_{2k} P(\partial_x)^{m-k} \frac{\partial^{2k}}{\partial t^{2k}}, \quad x \in \mathbb{R}^n, t \in \overline{\mathbb{R}_+^1}, \forall m \in \mathbb{N}_+.$$

Where $P(\partial_x)$ be a partial differential operator of any order, then we have

$$\begin{cases} P(\partial_x, \partial_t)u = f(x, t), & x \in \mathbb{R}^n, t \in \overline{\mathbb{R}_+^1}, \forall m \in \mathbb{N}_+, \\ \frac{\partial^r u}{\partial t^r} \Big|_{t=0} = \varphi_r(x), & r = 0, 1, 2, \dots, 2m-1. \end{cases} \quad (81)$$

$$\begin{aligned}
u(x, t) &= \int_0^t \int_0^{t-\tau} \frac{(t-\tau-\tau')^{2m-3}}{(2m-3)!} \sum_{j=1}^m \frac{a_j^{2m-2}}{\prod_{\substack{i=1 \\ i \neq j}}^m (a_j^2 - a_i^2)} \frac{\sinh(\tau' a_j P(\partial_x)^{1/2})}{a_j P(\partial_x)^{1/2}} f(x, \tau) d\tau' d\tau \\
&+ \sum_{k=1}^m b_{2k} P(\partial_x)^{m-k} \sum_{r=0}^{2k-1} \frac{\partial^{2k-1-r}}{\partial t^{2k-1-r}} \int_0^t \frac{(t-\tau)^{2m-3}}{(2m-3)!} \\
&\times \sum_{j=1}^m \frac{a_j^{2m-2}}{\prod_{\substack{i=1 \\ i \neq j}}^m (a_j^2 - a_i^2)} \frac{\sinh(\tau a_j P(\partial_x)^{1/2})}{a_j P(\partial_x)^{1/2}} \varphi_r(x) d\tau.
\end{aligned} \tag{82}$$

Let us prove these two theorems. According to the Theorem BI3 and Theorem BI4, we just need to prove the following Corollary of Theorem 8 and Theorem 9:

Corollary 8. Let a_1, a_2, \dots, a_m be arbitrary real or complex roots different from each other for $b_0 + b_1\chi + b_2\chi^2 + \dots + b_m\chi^m = 0$, and $P(\partial_x, \partial_t)$ be a partial differential operators defined by

$$P(\partial_x, \partial_t) = \sum_{k=0}^m b_k P(\partial_x)^{m-k} \frac{\partial^k}{\partial t^k}, \quad x \in \mathbb{R}^n, \quad t \in \overline{\mathbb{R}_+^1}, \quad \forall m \in \mathbb{N}_+.$$

Where $P(\partial_x)$ be a partial differential operator of any order, then we have

$$\begin{cases} P(\partial_x, \partial_t)u = 0, & x \in \mathbb{R}^n, \quad t \in \overline{\mathbb{R}_+^1}, \quad \forall m \in \mathbb{N}_+, \\ \left. \frac{\partial^r u}{\partial t^r} \right|_{t=0} = \varphi_r(x), & r = 0, 1, 2, \dots, m-1. \end{cases} \tag{83}$$

$$u(x, t) = \sum_{k=1}^m b_k P(\partial_x)^{m-k} \sum_{r=0}^{k-1} \frac{\partial^{k-1-r}}{\partial t^{k-1-r}} \int_0^t \frac{(t-\tau)^{m-2}}{(m-2)!} \sum_{j=1}^m \frac{a_j^{m-1} e^{\tau a_j P(\partial_x)}}{\prod_{\substack{i=1 \\ i \neq j}}^m (a_j - a_i)} \varphi_r(x) d\tau. \tag{84}$$

Proof. Considering initial conditions, the Laplace transform of the Eq (83) with respect to t is

$$\sum_{k=0}^m b_k P(\partial_x)^{m-k} \left(s^k U(x, s) - \sum_{r=0}^{k-1} s^{k-1-r} \varphi_r(x) \right) = 0.$$

Where $U(x, s) = \mathcal{L}u(x, t)$, considering $\prod_{i=1}^m (s - a_i P(\partial_x)) = \sum_{k=0}^m b_k s^k P(\partial_x)^{m-k}$ we have

$$\prod_{i=1}^m (s - a_i P(\partial_x)) U(x, s) - \sum_{k=1}^m b_k P(\partial_x)^{m-k} \sum_{r=0}^{k-1} s^{k-1-r} \varphi_r(x) = 0.$$

We need to introduce an abstract operators $G_m(\partial_x, t)$, defined as

$$G_m(\partial_x, t) = \mathcal{L}^{-1} \frac{1}{\prod_{i=1}^m (s - a_i P(\partial_x))}.$$

By solving $U(x, s)$, we have its inverse Laplace transform:

$$\begin{aligned}
u(x, t) &= \mathcal{L}^{-1} U(x, s) = \sum_{k=1}^m b_k P(\partial_x)^{m-k} \sum_{r=0}^{k-1} \mathcal{L}^{-1} \frac{s^{k-1-r}}{\prod_{i=1}^m (s - a_i P(\partial_x))} \varphi_r(x) \\
&= \sum_{k=1}^m b_k P(\partial_x)^{m-k} \sum_{r=0}^{k-1} \frac{\partial^{k-1-r}}{\partial t^{k-1-r}} G_m(\partial_x, t) \varphi_r(x).
\end{aligned} \tag{85}$$

Now let us solve $G_m(\partial_x, t)$. Considering initial conditions, the Laplace transform of the Eq (75) with respect to t is

$$\prod_{i=1}^m (s - a_i P(\partial_x)) U(x, s) = F(x, s),$$

where $F(x, s) = \mathcal{L}f(x, t)$. By solving $U(x, s)$ and using the convolution theorem, we have its inverse Laplace transform:

$$u(x, t) = \mathcal{L}^{-1} U(x, s) = \mathcal{L}^{-1} \frac{1}{\prod_{i=1}^m (s - a_i P(\partial_x))} F(x, s) = G_m(\partial_x, t) * f(x, t).$$

By comparing (76) with $u(x, t) = G_m(\partial_x, t) * f(x, t)$, we have the expression of the abstract operators $G_m(\partial_x, t)$:

$$G_m(\partial_x, t) = \int_0^t \frac{(t - \tau)^{m-2}}{(m-2)!} \sum_{j=1}^m \frac{a_j^{m-1}}{\prod_{i=1, i \neq j}^m (a_j - a_i)} e^{\tau a_j P(\partial_x)} d\tau. \quad (86)$$

Applying (86) to (85), thus the Corollary 8 is proved.

Corollary 9. Let a_1, a_2, \dots, a_m be arbitrary real or complex roots different from each other, which satisfy $\sum_{k=0}^m b_{2k} \chi^{2k} = \prod_{i=1}^m (\chi^2 - a_i^2)$, and $P(\partial_x, \partial_t)$ be a partial differential operator defined by

$$P(\partial_x, \partial_t) = \sum_{k=0}^m b_{2k} P(\partial_x)^{m-k} \frac{\partial^{2k}}{\partial t^{2k}}, \quad x \in \mathbb{R}^n, \quad t \in \overline{\mathbb{R}_+^1}, \quad \forall m \in \mathbb{N}_+.$$

Where $P(\partial_x)$ is a partial differential operator of any order, then we have

$$\begin{cases} P(\partial_x, \partial_t)u = 0, & x \in \mathbb{R}^n, \quad t \in \overline{\mathbb{R}_+^1}, \quad \forall m \in \mathbb{N}_+, \\ \left. \frac{\partial^r u}{\partial t^r} \right|_{t=0} = \varphi_r(x), & r = 0, 1, 2, \dots, 2m-1. \end{cases} \quad (87)$$

$$\begin{aligned} u(x, t) &= \sum_{k=1}^m b_{2k} P(\partial_x)^{m-k} \sum_{r=0}^{2k-1} \frac{\partial^{2k-1-r}}{\partial t^{2k-1-r}} \int_0^t \frac{(t - \tau)^{2m-3}}{(2m-3)!} \\ &\quad \times \sum_{j=1}^m \frac{a_j^{2m-2}}{\prod_{i=1, i \neq j}^m (a_j^2 - a_i^2)} \frac{\sinh(\tau a_j P(\partial_x)^{1/2})}{a_j P(\partial_x)^{1/2}} \varphi_r(x) d\tau. \end{aligned} \quad (88)$$

Proof. Considering initial conditions, the Laplace transform of the Eq (87) with respect to t is

$$\sum_{k=0}^m b_{2k} P(\partial_x)^{m-k} \left(s^{2k} U(x, s) - \sum_{r=0}^{2k-1} s^{2k-1-r} \varphi_r(x) \right) = 0.$$

Where $U(x, s) = \mathcal{L}u(x, t)$, considering $\prod_{i=1}^m (s^2 - a_i^2 P(\partial_x)) = \sum_{k=0}^m b_{2k} s^{2k} P(\partial_x)^{m-k}$ we have

$$\prod_{i=1}^m (s^2 - a_i^2 P(\partial_x)) U(x, s) - \sum_{k=1}^m b_{2k} P(\partial_x)^{m-k} \sum_{r=0}^{2k-1} s^{2k-1-r} \varphi_r(x) = 0.$$

We need to introduce an abstract operators $G_m(\partial_x, t)$, defined as

$$G_m(\partial_x, t) = \mathcal{L}^{-1} \frac{1}{\prod_{i=1}^m (s^2 - a_i^2 P(\partial_x))}.$$

By solving $U(x, s)$, we have its inverse Laplace transform:

$$\begin{aligned} u(x, t) &= \mathcal{L}^{-1}U(x, s) = \sum_{k=1}^m b_{2k} P(\partial_x)^{m-k} \sum_{r=0}^{2k-1} \mathcal{L}^{-1} \frac{s^{2k-1-r}}{\prod_{i=1}^m (s^2 - a_i^2 P(\partial_x))} \varphi_r(x) \\ &= \sum_{k=1}^m b_{2k} P(\partial_x)^{m-k} \sum_{r=0}^{2k-1} \frac{\partial^{2k-1-r}}{\partial t^{2k-1-r}} G_m(\partial_x, t) \varphi_r(x). \end{aligned} \quad (89)$$

Now let us solve the $G_m(\partial_x, t)$. Considering initial conditions, the Laplace transform of the Eq (77) with respect to t is

$$\prod_{i=1}^m (s^2 - a_i^2 P(\partial_x)) U(x, s) = F(x, s),$$

where $F(x, s) = \mathcal{L}f(x, t)$. By solving $U(x, s)$ and using the convolution theorem, we have its inverse Laplace transform:

$$u(x, t) = \mathcal{L}^{-1}U(x, s) = \mathcal{L}^{-1} \frac{1}{\prod_{i=1}^m (s^2 - a_i^2 P(\partial_x))} F(x, s) = G_m(\partial_x, t) * f(x, t).$$

By comparing (78) with $u(x, t) = G_m(\partial_x, t) * f(x, t)$, we have the expression of the abstract operators $G_m(\partial_x, t)$:

$$G_m(\partial_x, t) = \int_0^t \frac{(t-\tau)^{2m-3}}{(2m-3)!} \sum_{j=1}^m \frac{a_j^{2m-2}}{\prod_{i=1, i \neq j}^m (a_j^2 - a_i^2)} \frac{\sinh(\tau a_j P(\partial_x)^{1/2})}{a_j P(\partial_x)^{1/2}} d\tau. \quad (90)$$

Applying (90) to (89), thus the Corollary 9 is proved.

3.2 Analytic solution of Cauchy problem

Theorem 10. Let $\Delta_n = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}$ be an n -dimensional Laplacian. If $n - 2 = 2\nu + 1$, $\nu \in \mathbb{N}$, then $\forall f(x, \tau) \in C^{2\nu}(\mathbb{R}^n)$, and $\forall \varphi_j(x) \in C^{2m+2\nu-2}(\mathbb{R}^n)$, we have

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - a^2 \Delta_n \right)^m u = f(x, t), & x \in \mathbb{R}^n, t \in \overline{\mathbb{R}_+^1}, \forall m \in \mathbb{N}_+, \\ \frac{\partial^j u}{\partial t^j} \Big|_{t=0} = \varphi_j(x), & j = 0, 1, 2, \dots, 2m-1. \end{cases} \quad (91)$$

$$\begin{aligned} u(x, t) &= \int_0^t d\tau \int_0^{t-\tau} d\tau' \frac{[(t-\tau)^2 - \tau'^2]^{m-2} \tau'^2}{(2m-2)!! (2m-4)!!} \\ &\quad \times \underbrace{\int_0^{\tau'} \tau' d\tau' \dots \int_0^{\tau'} \frac{(a^2 \Delta_n)^\nu}{S'_n} \int_{S'_n} f(\xi', \tau) dS'_n \tau' d\tau'}_\nu \\ &\quad + \frac{1}{(2m-2)!!} \sum_{r=0}^{\nu-1} \int_0^t \frac{(t-\tau)^{2m+2r-1}}{(2m+2r-1)!! (2r)!!} (a^2 \Delta_n)^r f(x, \tau) d\tau \\ &\quad + \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} (a^2 \Delta_n)^{k+\nu} \sum_{j=0}^{2m-1-2k} \frac{\partial^{2m-1-2k-j}}{\partial t^{2m-1-2k-j}} \int_0^t d\tau \frac{(t^2 - \tau^2)^{m-2} \tau^2}{(2m-2)!! (2m-4)!!} \end{aligned}$$

$$\begin{aligned}
& \times \underbrace{\int_0^\tau \tau d\tau \cdots \int_0^\tau \frac{1}{S_n} \int_{S_n} \varphi_j(\xi) dS_n \tau d\tau}_{\nu} + \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} \sum_{j=0}^{2m-1-2k} \\
& \times \sum_{i=0}^{\nu-1} \binom{m-1+i}{i} \frac{t^{2k+2i+j}}{(2k+2i+j)!} (a^2 \Delta_n)^{k+i} \varphi_j(x). \tag{92}
\end{aligned}$$

Where $n-2 = 2\nu+1$, $S'_n = 2(2\pi)^{\nu+1}(a\tau')^{n-1}$, $S_n = 2(2\pi)^{\nu+1}(a\tau)^{n-1}$, and $\xi' \in \mathbb{R}_n$ is the integral variable. The integral is on the hypersphere $(\xi'_1 - x_1)^2 + (\xi'_2 - x_2)^2 + \cdots + (\xi'_n - x_n)^2 = (a\tau')^2$, and dS'_n is its surface element. $\xi \in \mathbb{R}_n$ is the integral variable on the hypersphere $(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + \cdots + (\xi_n - x_n)^2 = (a\tau)^2$, and dS_n is its surface element.

Proof. According to (64), we can easily derive:

$$\begin{aligned}
\frac{\sinh(at\Delta_n^{1/2})}{a\Delta_n^{1/2}} f(x) &= t \underbrace{\int_0^t t dt \cdots \int_0^t}_{\nu} \frac{(a^2 \Delta_n)^\nu}{S_n} \int_{S_n} f(\xi) dS_n t dt \\
&+ \sum_{i=0}^{\nu-1} \frac{t^{2i+1}}{(2i+1)!} (a^2 \Delta_n)^i f(x), \quad \forall f(x) \in C^{2\nu}(\mathbb{R}^n). \tag{93}
\end{aligned}$$

Where Δ_n is an n -dimensional Laplacian, $n-2 = 2\nu+1$, $S_n = 2(2\pi)^{\nu+1}(at)^{n-1}$. $\xi \in \mathbb{R}_n$ is the integral variable on the hypersphere $(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + \cdots + (\xi_n - x_n)^2 = (at)^2$, and dS_n is its surface element.

In Theorem 7, let $P(\partial_x) = a^2 \Delta_n$, then Theorem 10 is proved by the substitution of (93).

Theorem 11. Let a_1, a_2, \dots, a_m be arbitrary real roots different from each other, which satisfy $\sum_{k=0}^m b_{2k} \chi^{2k} = \prod_{i=1}^m (\chi^2 - a_i^2)$, and $P(\partial_x, \partial_t)$ be a partial differential operator defined by

$$P(\partial_x, \partial_t) = \sum_{k=0}^m b_{2k} \Delta_n^{m-k} \frac{\partial^{2k}}{\partial t^{2k}}, \quad x \in \mathbb{R}^n, t \in \overline{\mathbb{R}_+^1}, \forall m \in \mathbb{N}_+.$$

Where Δ_n is an n -dimensional Laplacian. If $n-2 = 2\nu+1$, $\nu = 0, 1, 2, \dots$, then $\forall \varphi_r(x) \in C^{2m+2\nu-2}(\mathbb{R}^n)$, and $\forall f(x, \tau) \in C^{2\nu}(\mathbb{R}^n)$, we have

$$\begin{cases} P(\partial_x, \partial_t)u = f(x, t), & x \in \mathbb{R}^n, t \in \overline{\mathbb{R}_+^1}, \forall m \in \mathbb{N}_+, \\ \left. \frac{\partial^r u}{\partial t^r} \right|_{t=0} = \varphi_r(x), & r = 0, 1, 2, \dots, 2m-1. \end{cases} \tag{94}$$

$$\begin{aligned}
u(x, t) &= \int_0^t d\tau \int_0^{t-\tau} d\tau' \frac{(t-\tau-\tau')^{2m-3}}{(2m-3)!} \sum_{j=1}^m \frac{a_j^{2m-2}}{\prod_{i \neq j}^m (a_j^2 - a_i^2)} \\
&\times \left(\underbrace{\tau' \int_0^{\tau'} \tau' d\tau' \cdots \int_0^{\tau'} \frac{(a_j^2 \Delta_n)^\nu}{S'_{n,j}}}_{\nu} \int_{S'_{n,j}} f(\xi', \tau) dS'_{n,j} \tau' d\tau' + \sum_{l=0}^{\nu-1} \frac{a_j^{2l} \tau'^{2l+1}}{(2l+1)!} \Delta_n^l f(x, \tau) \right) \\
&+ \sum_{k=1}^m b_{2k} \Delta_n^{m-k} \sum_{r=0}^{2k-1} \frac{\partial^{2k-1-r}}{\partial t^{2k-1-r}} \int_0^t d\tau \frac{(t-\tau)^{2m-3}}{(2m-3)!} \sum_{j=1}^m \frac{a_j^{2m-2}}{\prod_{i \neq j}^m (a_j^2 - a_i^2)}
\end{aligned}$$

$$\times \left(\tau \underbrace{\int_0^\tau \tau d\tau \cdots \int_0^\tau}_{\nu} \frac{(a_j^2 \Delta_n)^\nu}{S_{n,j}} \int_{S_{n,j}} \varphi_r(\xi) dS_{n,j} \tau d\tau + \sum_{l=0}^{\nu-1} \frac{a_j^{2l} \tau^{2l+1}}{(2l+1)!} \Delta_n^l \varphi_r(x) \right). \quad (95)$$

Where $S'_{n,j} = 2(2\pi)^{\nu+1}(a_j \tau')^{n-1}$, $S_{n,j} = 2(2\pi)^{\nu+1}(a_j \tau)^{n-1}$, and $\xi' \in \mathbb{R}_n$ is the integral variable. The integral is on the hypersphere $(\xi'_1 - x_1)^2 + (\xi'_2 - x_2)^2 + \cdots + (\xi'_n - x_n)^2 = (a_j \tau')^2$, and $dS'_{n,j}$ is its surface element. $\xi \in \mathbb{R}_n$ is the integral variable on the hypersphere $(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + \cdots + (\xi_n - x_n)^2 = (a_j \tau)^2$, and $dS_{n,j}$ is its surface element.

Proof. In Theorem 9, let $P(\partial_x) = \Delta_n$, then Theorem 11 is proved by the substitution of (93).

Similarly, we can easily obtain explicit solutions of the Cauchy problem of more complex partial differential equations. Therefore, using the method of abstract operators, we have established a general theory of initial value problems for linear higher-order partial differential equations.

3.3 Initial-boundary value problem and Hilbert space

Clearly, similar to $\sinh(h\partial_x)$ and $\cosh(h\partial_x)$, the abstract operators

$$\frac{\sinh(at\Delta_n^{1/2})}{a\Delta_n^{1/2}} \quad \text{and} \quad \cos(at\Delta_n^{1/2}) = \frac{\partial}{\partial t} \frac{\sinh(at\Delta_n^{1/2})}{a\Delta_n^{1/2}}$$

on the left side of (93) are bounded operators in a Hilbert space, and can act on the whole Hilbert space. Thus we can attach proper boundary conditions to the initial value problem introduced by Theorem 7, which makes the given functions $f(x, t)$, $\varphi_j(x)$ in Eq (91) and (94) become functions with boundary conditions. Then solving this initial-boundary value problem comes down to expressing $f(x, t)$, $\varphi_j(x)$ in a Hilbert space H within the given domain Ω .

For the initial-boundary value problem, the operator $P(\partial_x)$ must have the characteristic function related to boundary conditions, in order to expand the known function $f(x, \tau)$, $\varphi_r(x)$ in (71), (80) and (82) by using the characteristic function of $P(\partial_x)$. $P(\partial_x)$ in Theorem 7, Theorem 8 and Theorem 9 can be variable-coefficient partial differential operators. For instance, if $P(\partial_x)$ is a self-adjoint operator defined on a Hilbert space H , then the abstract operators

$$\frac{\sinh(tP(\partial_x)^{1/2})}{P(\partial_x)^{1/2}} \quad \text{and} \quad \cosh(tP(\partial_x)^{1/2}) = \frac{\partial}{\partial t} \frac{\sinh(tP(\partial_x)^{1/2})}{P(\partial_x)^{1/2}}$$

are the continuous operators on the Hilbert space. In this case, we can attach proper boundary conditions to the initial value problems in (70), (79) and (81). Therefore, the given function $f(x, t)$, $\varphi_r(x)$ becomes a function with boundary conditions, and can be expressed in the Hilbert space H within the given domain Ω . In order to solve the corresponding initial-boundary value problem, we need to solve the characteristic value problem of $P(\partial_x)$ under given boundary conditions to determine a set of orthogonal functions, which generates a linear manifold of the Hilbert space, thus $f(x, t)$, $\varphi_r(x)$ can be expressed in the Hilbert space.

In (71), (80) and (82), if $f(x, \tau)$, $\varphi_r(x) \in L^2(\Omega)$, and $P(\partial_x)$ is a second-order linear self-adjoint elliptic operator, namely

$$P(\partial_x)u = \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + c(x)u, \quad x \in \Omega \subset \mathbb{R}^n, \quad (96)$$

then boundary conditions can be added for the definite solution problems (70), (79) and (81): $\overline{Bu}|_{\partial\Omega} = 0$ representing $u|_{\partial\Omega} = 0$ or

$$\left[\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \cos\langle \mathbf{a}, x_i \rangle + b(x)u \right]_{\partial\Omega} = 0.$$

Where \mathbf{a} is the unit outward normal of $\partial\Omega$. Thus this kind of initial boundary value problems boils down to solving the characteristic value problem of the first boundary value problem of second-order linear self-adjoint elliptic operator:

$$\begin{cases} \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + c(x)u = -\lambda u, & x \in \Omega \subset \mathbb{R}^n, \\ \overline{Bu}|_{\partial\Omega} = 0 \end{cases} \quad (97)$$

Reference [7] quotes the following theorem:

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain, and $\partial\Omega$ be smooth. Let $a_{ij} = a_{ji}$, and there exists $\theta > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2, \quad x \in \Omega.$$

And let $a_{ij} \in C^1(\overline{\Omega})$, $c(x) \in C(\overline{\Omega})$, $b(x) \in C(\partial\Omega)$, then (97) has the following countable characteristic values:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\nu \leq \cdots, \quad \lim_{\nu \rightarrow \infty} \lambda_\nu = \infty$$

(If $(a_{ij}) = I$ is a unit matrix, then $\lambda_1 = 0$ when $b(x) = c(x) = 0$. When $b(x) \geq 0$, $c(x) \geq 0$ and one of them does not identically equal to zero, $\lambda_1 > 0$) and the corresponding characteristic functions $e_1(x), e_2(x), \dots, e_\nu(x), \dots$, satisfy

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial e_i}{\partial x_i} \right) + c(x)e_i = -\lambda_i e_i, \quad (e_i, e_j) = \delta_{ij} \quad (98)$$

and $\{e_j(x)\}_{j=1}^\infty$ is complete in $L^2(\Omega)$, that is for an arbitrary $f(x) \in L^2(\Omega)$, there exists c_j such that

$$\lim_{m \rightarrow \infty} \|f - \sum_{j=1}^m c_j e_j\|_{L^2(\Omega)} = 0.$$

Therefore, for (71), (80) and (82), by $f(x, \tau), \varphi_r(x) \in L^2(\Omega)$ we also have

$$\lim_{m \rightarrow \infty} \|f(x, \tau) - \sum_{j=1}^m c_j(\tau) e_j(x)\|_{L^2(\Omega)} = 0.$$

$$\lim_{m \rightarrow \infty} \|\varphi_r(x) - \sum_{j=1}^m c_j e_j(x)\|_{L^2(\Omega)} = 0.$$

Clearly, based on (96) and (98), we have the extension of abstract operators in Hilbert space:

$$e^{\tau a_j P(\partial_x)} e_i(x) = e^{-\tau a_j \lambda_i} e_i(x), \quad \frac{\sinh(\tau a_j P(\partial_x)^{1/2})}{a_j P(\partial_x)^{1/2}} e_i(x) = \frac{\sin(a_j \sqrt{\lambda_i} \tau)}{a_j \sqrt{\lambda_i}} e_i(x). \quad (99)$$

Where $x \in \Omega \subset \mathbb{R}^n$, $j = 1, 2, \dots, m$. Then we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \|e^{\tau' a_j P(\partial_x)} f(x, \tau) - \sum_{j=1}^m (f, e_j) e^{-\tau' a_j \lambda_i} e_j(x)\|_{L^2(\Omega)} &= 0. \\ \lim_{m \rightarrow \infty} \|e^{\tau a_j P(\partial_x)} \varphi_r(x) - \sum_{j=1}^m (\varphi_r, e_j) e^{-\tau a_j \lambda_i} e_j(x)\|_{L^2(\Omega)} &= 0. \\ \lim_{m \rightarrow \infty} \left\| \frac{\sinh(\tau' a_j P(\partial_x)^{1/2})}{a_j P(\partial_x)^{1/2}} f(x, \tau) - \sum_{j=1}^m (f, e_j) \frac{\sin(a_j \sqrt{\lambda_i} \tau')}{a_j \sqrt{\lambda_i}} e_j(x) \right\|_{L^2(\Omega)} &= 0. \\ \lim_{m \rightarrow \infty} \left\| \frac{\sinh(\tau a_j P(\partial_x)^{1/2})}{a_j P(\partial_x)^{1/2}} \varphi_r(x) - \sum_{j=1}^m (\varphi_r, e_j) \frac{\sin(a_j \sqrt{\lambda_i} \tau)}{a_j \sqrt{\lambda_i}} e_j(x) \right\|_{L^2(\Omega)} &= 0. \end{aligned}$$

Thus when (70), (79) and (81) satisfy the boundary condition $\overline{B}u|_{\partial\Omega} = 0$, if the orthogonal function system $\{e_j(x)\}_{j=1}^\infty$ is known, we have the solution of the corresponding initial boundary value problem.

In conclusion, based on the method of abstract operators and comprehensively applying the Laplace transform and Hilbert space theory, we establish a general theory of initial value and initial boundary value problems of linear partial differential equations, and illustrate broad application prospects of the abstract operator theory, which will definitely have a wide and profound influence on various scientific fields.

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